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Traveling Wave Solutions of the Quintic Complex One-Dimensional Ginzburg-Landau Equation

Hans Werner Schürmann[1], Valery Serov[2]

[1]Department of Physics, University of Osnabrück, Osnabrück, Germany
[2]Research Unit of Mathematical Sciences, University of Oulu, Finland and Moscow Centre of Fundamental and Applied Mathematics—MSU, Moscow, Russia
Email: hwschuer@uos.de, vserov@cc.oulu.fi

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Abstract

A subset of traveling wave solutions of the quintic complex Ginzburg-Landau equation (QCGLE) is presented in compact form. The approach consists of the following parts: 1) Reduction of the QCGLE to a system of two ordinary differential equations (ODEs) by a traveling wave ansatz; 2) Solution of the system for two (ad hoc) cases relating phase and amplitude; 3) Presentation of the solution for both cases in compact form; 4) Presentation of constraints for bounded and for singular positive solutions by analysing the analytical properties of the solution by means of a phase diagram approach. The results are exemplified numerically.

Keywords

Ginzburg-Landau Equation, Weierstrass' Elliptic Function, Phase Diagram

1. Introduction

As a partial differential equation (PDE) the quintic complex Ginzburg-Landau equation (QCGLE) is one of the most studied nonlinear equations in physics. Apart from many applications in the natural sciences [1] [2], the equation is interesting in and of itself: as a nonintegrable, nonlinear PDE; it admits a reduction to an autonomous ODE described in the simplest case by introducing the new variable $z = x - ct$, where x is a space coordinate and t is time (traveling wave reduction). While integrable PDEs are "easy" to solve by the Inverse Scattering Transform, for nonintegrable PDEs no such method is known to obtain solutions.

Probably, due to this reason, several non-perturbative methods have been proposed to find some particular solutions of (partly) nonintegrable systems ("tanh-method", "exponential-method", "Riccati-method", "Jacobi expansion-method", ..., see also [3] and references therein). A comprehensive treatment of the QCGLE using Panlevé analysis is presented in [4] [5] [6] [7]. In particular, an algorithm able to provide in closed form all those traveling wave solutions that are elliptic or degenerate elliptic has been applied to the QCGLE (results in [4], Equation (52) and in [5], Section 6).

As a starting point for perturbation calculations or stability analysis exact traveling solutions of nonlinear equations such as (2) below are very useful, however rare. Remarkably, if certain constraints for the parameters are satisfied, solutions can be derived. In [8] [9] [10], special relations between phase and amplitude of the traveling wave are assumed in the solution ansatz, leading to particular analytical solutions. Following this path, we propose a relatively simple (ad hoc) approach which allows us to find exact analytical solutions not known in the relevant literature.

The rest of the paper is organised as follows. In Section 2 we reduce the QCGLE to a system of two ordinary differential equations, specify it for two particular (ad hoc) relations between phase function and amplitude, and describe the procedure to obtain the ansatz parameters c, ω, d, b_0, b_1 and constraints of these relations. An exact solution together with its dependence on the parameters of the QCGLE is presented in Section 3. For elucidation examples are presented in Section 4. In Section 5, the paper concludes with comments on articles having a certain contact to the present paper and with suggestions for further investigations.

2. Reduction of QCGLE to First Order Nonlinear ODEs

We seek particular (traveling wave) solutions [2] [8] [9] [10] [11]

$$\psi(x,t) = f(x-ct)\mathrm{e}^{i(\phi(x-ct)-\omega t)}, \quad c \in \mathbf{R}, \quad \omega \in \mathbf{R}, \tag{1}$$

where f and ϕ are real-valued, of the QCGLE

$$\begin{aligned} & i\frac{\partial}{\partial t}\psi(x,t) + (c_3 + ih_3)|\psi(x,t)|^2 \psi(x,t) + (c_5 + ih_5)|\psi(x,t)|^4 \psi(x,t) \\ & + (c_1 - ih_1)\frac{\partial^2}{\partial x^2}\psi(x,t) - i\varepsilon\psi(x,t) = 0, \end{aligned} \tag{2}$$

with complex-valued function $\psi(x,t)$ and with real dimensionless constants $\varepsilon, h_1, h_3, h_5, c_1, c_3, c_5$. As mentioned in the Introduction, Equation (2) has a wide range of applications. Thus the meaning of variables and parameters may be quite different (see [1] [8] and references therein). Usually, x denotes the scaled propagation distance and t the scaled time.

To find real f, ϕ, c, ω, together with corresponding assumptions and constraints for their existence, we insert ansatz (1) into Equation (2), set $z = x - ct$, introduce $F(z) = f^2(z)$, $\tau(z) = \phi'(z)$, and separate real and imaginary parts.

Hence we obtain the system of two ordinary differential equations for functions $F(z)$ and $\tau(z)$

$$\begin{aligned}&4\omega F^2(z)+4c_3F^3(z)+4c_5F^4(z)+4c\tau(z)F^2(z)-4c_1\tau^2(z)F^2(z)\\&+4h_1\tau(z)F(z)F'(z)-c_1(F'(z))^2+4h_1\tau'(z)F^2(z)+2c_1F(z)F''(z)=0,\end{aligned} \tag{3}$$

$$\begin{aligned}&-4\varepsilon F^2(z)+4h_3F^3(z)+4h_5F^4(z)+4h_1\tau^2(z)F^2(z)-2cF(z)F'(z)\\&+4c_1\tau(z)F(z)F'(z)+h_1(F'(z))^2+4c_1\tau'(z)F^2(z)-2h_1F(z)F''(z)=0.\end{aligned} \tag{4}$$

As mentioned above, there is a general method in the literature [5] [12] to find all elliptic solutions of the QCGLE. We note however that the explicit solution obtained by this method (see Equation (52) in [4b]) exists at the price of 5 constraints among the coefficients of the system (3)-(4). Thus it is obvious to try for a different approach. In particular, we seek for solutions $F(z)$ of the system (3)-(4) by assuming [8] [9] [10] two possibilities for $\tau(z)$

$$\tau(z)=d\frac{F'(z)}{F(z)}, \tag{5}$$

$$\tau(z)=b_0+b_1F(z), \tag{6}$$

where real constants d,b_0,b_1 are to be determined.

We consider the solutions $F(z)$ of the system (3)-(4), with assumptions for $\tau(z)$ from above, and determine parameters c,ω,d,b_0,b_1 for both cases (5), (6) separately. In each part (for (5) and for (6)) we derive the solution $\psi(x,t)$ and study its properties (disregarding its stability). We note that system (3)-(4) is equivalent to (13)-(14) in [5].

To simplify formulas we use the abbreviations

$$\begin{aligned}&D_1=c_1^2+h_1^2,\quad D_2=c_3h_1+c_1h_3,\quad D_3=c_1c_3-h_1h_3,\\&D_4=c_5h_1+c_1h_5,\quad D_5=c_1c_5-h_1h_5\end{aligned}$$

in what follows.

First, we consider Equations (3), (4) subject to $\tau(z)=d\frac{F'(z)}{F(z)}$. Equations (3), (4) read in this case

$$\begin{aligned}&4\omega F^2(z)+4c_3F^3(z)+4c_5F^4(z)+4cdF(z)F'(z)\\&-c_1(1+4d^2)(F'(z))^2+(4h_1d+2c_1)F(z)F''(z)=0,\end{aligned} \tag{7}$$

$$\begin{aligned}&-4\varepsilon F^2(z)+4h_3F^3(z)+4h_5F^4(z)-2cF(z)F'(z)\\&+h_1(1+4d^2)(F'(z))^2+(4c_1-2h_1)F(z)F''(z)=0.\end{aligned} \tag{8}$$

By eliminating $F''(z)$ from (7) and (8), we obtain for $F'(z)$

$$F'(z)=F(z)\left(\pm\sqrt{a+bF(z)+fF^2(z)}+\frac{cc_1}{2dD_1}\right), \tag{9}$$

with

$$a=\frac{c^2c_1^2}{4d^2D_1^2}+\frac{2\varepsilon(c_1+2dh_1)+2\omega(2dc_1-h_1)}{d(1+4d^2)D_1}, \tag{10}$$

$$b=\frac{4dD_3-2D_2}{d\left(1+4d^2\right)D_1}, \tag{11}$$

$$f=\frac{4dD_5-2D_4}{d\left(1+4d^2\right)D_1}. \tag{12}$$

Using (9), we get

$$F''(z)=F'(z)\left(\pm\frac{2a+3bF(z)+4fF^2(z)}{2\sqrt{a+bF(z)+fF^2(z)}}+\frac{cc_1}{2dD_1}\right). \tag{13}$$

Substituting (9) and (13) to the system (7)-(8), we obtain the following equivalent system only in terms of $F(z)$ and the parameters of QCGLE

$$\begin{aligned}&4acdD_1\left(2h_1d+c_1\right)^2+\Big[c^2c_1\left(c_1^2+4d^2\left(c_1^2+2h_1^2\right)+4c_1h_1d\right)\\&+4ad^2D_1^2\left(c_1-4c_1d^2+4h_1d\right)+16\omega d^2D_1^2\Big]\sqrt{a+bF(z)+fF^2(z)}\\&+F(z)\Big[2bcdD_1\left(8h_1^2d^2+10c_1h_1d+3c_1^2\right)\\&+8d^2D_1^2\left(2c_3+b\left(c_1-2c_1d^2+3h_1d\right)\right)\sqrt{a+bF(z)+fF^2(z)}\Big]\\&+F^2(z)\Big[8cdD_1f\left(c_1^2+3c_1h_1d+2h_1^2d^2\right)\\&+4d^2D_1^2\left(4c_5+f\left(3c_1-4c_1d^2+8h_1d\right)\right)\sqrt{a+bF(z)+fF^2(z)}\Big]=0\end{aligned} \tag{14}$$

and

$$\begin{aligned}&4acdD_1\left(4c_1h_1d^2+2\left(c_1^2-h_1^2\right)d-c_1h_1\right)+\Big[c^2c_1\left(4c_1h_1d^2-4h_1^2d-c_1h_1\right)\\&+4ad^2D_1^2\left(4h_1d^2+4c_1d-h_1\right)-16\varepsilon d^2D_1^2\Big]\sqrt{a+bF(z)+fF^2(z)}\\&+F(z)\Big[2bcdD_1\left(8c_1^2d^2+\left(c_1^2-4h_1^2\right)d-3c_1h_1\right)\\&+8d^2D_1^2\left(2h_3+b\left(3c_1d-h_1+2h_1d^2\right)\right)\sqrt{a+bF(z)+fF^2(z)}\Big]\\&+F^2(z)\Big[8cdD_1f\left(2c_1h_1d+\left(2c_1^2-h_1^2\right)d-c_1h_1\right)\\&+4d^2D_1^2\left(4h_5+f\left(8c_1d+4h_1d^2-3h_1\right)\right)\sqrt{a+bF(z)+fF^2(z)}\Big]=0\end{aligned} \tag{15}$$

In Equations (14)-(15) the coefficients in front of $F(z)$, $F^2(z)$, $\sqrt{a+bF(z)+fF^2(z)}$ as well as free coefficients must be equal to zero, leading to equations which imply $c=0$ necessarily. With $c=0$ the system (14)-(15) can be simplified to the following system of six equations:

$$4\omega=a\left(4c_1d^2-4h_1d-c_1\right),\quad 4\varepsilon=a\left(4h_1d^2+4c_1d-h_1\right), \tag{16}$$

$$b\left(2c_1d^2-3h_1d-c_1\right)=2c_3,\quad b\left(2h_1d^2+3c_1d-h_1\right)=-2h_3, \tag{17}$$

$$f\left(4c_1d^2-8h_1d-3c_1\right)=4c_5,\quad f\left(4h_1d^2+8c_1d-3h_1\right)=-4h_5. \tag{18}$$

Combining the first Equation (16) and (10), we obtain

$$\omega=\frac{\varepsilon\left(4c_1d^2-4h_1d-c_1\right)}{4h_1d^2+4c_1d-h_1}. \tag{19}$$

Furthermore, Equations (16)-(18) are solved by

$$d=\frac{-3D_3\pm\sqrt{9D_3^2+8D_2^2}}{4D_2}\quad\text{and}\quad d=\frac{-2D_5\pm\sqrt{4D_5^2+3D_4^2}}{2D_4}. \tag{20}$$

Consistency of d's in the Equation (20) leads to the constraints

$$\frac{-3D_3\pm\sqrt{9D_3^2+8D_2^2}}{4D_2}=\frac{-2D_5\pm\sqrt{4D_5^2+3D_4^2}}{2D_4}, \tag{21}$$

necessary for the existence of solutions d.

Thus, parameters d,ω and a,b,f in Equation (9) are expressed according to (19)-(20) and (10)-(12), respectively in terms of the parameters of the QCGLE (with constraint (21)). The solution $F(z)$ of Equation (9) is presented below.

Second, considering the case $\tau(z)=b_0+b_1F(z)$, and inserting $\tau(z)$ into the system (3)-(4), we get

$$\begin{aligned}&4\left(\omega+cb_0-c_1b_0^2\right)F^2(z)+4\left(c_3+cb_1-2c_1b_0b_1\right)F^3(z)+4\left(c_5-c_1b_1^2\right)F^4(z)\\&-c_1\left(F'(z)\right)^2+4h_1\left(b_0F(z)+2b_1F^2(z)\right)F'(z)+2c_1F(z)F''(z)=0\end{aligned} \tag{22}$$

and

$$\begin{aligned}&4\left(h_1b_0^2-\varepsilon\right)F^2(z)+4\left(h_3+2h_1b_0b_1\right)F^3(z)+4\left(h_5+h_1b_1^2\right)F^4(z)+h_1\left(F'(z)\right)^2\\&+\left(\left(4c_1b_0-2c\right)F(z)+8c_1b_1F^2(z)\right)F'(z)-2h_1F(z)F''(z)=0.\end{aligned} \tag{23}$$

Eliminating $F''(z)$ from (22) and (23), $F'(z)$ can be derived as

$$F'(z)=\frac{\left(h_1\left(\omega+cb_0\right)-c_1\varepsilon\right)F(z)+\left(D_2+ch_1b_1\right)F^2(z)+D_4F^3}{\frac{cc_1}{2}-b_0D_1-2b_1D_1F(z)}. \tag{24}$$

Assuming for simplicity $b_0=\frac{cc_1}{2D_1}$, we get

$$F'(z)=-\frac{h_1\left(\omega+\frac{c^2c_1}{2D_1}\right)-\varepsilon c_1+\left(ch_1b_1+D_2\right)F(z)+D_4F^2(z)}{2b_1D_1} \tag{25}$$

and hence

$$F''(z)=-\frac{\left(ch_1b_1+D_4+2D_2F(z)\right)F'(z)}{2b_1D_1}. \tag{26}$$

Substitution of (25) and (26) into system (22)-(23) and considering vanishing coefficients of powers of $F(z)$, leads to

$$2D_1\left(h_1\omega-c_1\varepsilon\right)+c^2c_1h_1=0, \tag{27}$$

$$-16b_1^4c_1D_1^2+3c_1D_4^2+16c_1b_1^2D_1D_5=0, \tag{28}$$

$$c_1D_2D_4+4b_1^2c_1D_1D_3=0, \tag{29}$$

$$\begin{aligned}&-2cc_1h_1b_1D_1D_2+c_1\left(c^2c_1h_1D_4+D_1\left(D_2^2-2c_1\varepsilon D_4+2h_1\omega D_4\right)\right)\\&-b_1^2D_1\left(c^2c_1\left(4c_1^2-3h_1^2\right)+16D_1c_1\left(h_1\varepsilon+c_1\omega\right)\right)=0,\end{aligned} \tag{30}$$

where $c \neq 0, h_1 \neq 0, b_1 \neq 0$ have been assumed. Parameters ω, b_1 and c must satisfy system (27)-(30). Equation (27) implies

$$\omega = \frac{2c_1\varepsilon D_1 - c^2 c_1 h_1}{2h_1 D_1}. \tag{31}$$

Solutions of (28) and (29) are

$$b_1^2 = \frac{2D_5 + \sqrt{4D_5^2 + 3D_4^2}}{4D_1} \tag{32}$$

and

$$b_1^2 = -\frac{D_4 D_2}{4D_1 D_3}, \tag{33}$$

respectively. Inserting ω into (30) and solving for c we get

$$c = \frac{-h_1 D_2 \pm 2\sqrt{D_1 D_2^2 + 4\frac{\varepsilon}{h_1} b_1^2 D_1^2 \left(4c_1^2 + 3h_1^2\right)}}{b_1\left(3h_1^2 + 4c_1^2\right)}. \tag{34}$$

Consistency of (32) and (33) yields the constraint

$$D_4 D_2^2 + 4D_2 D_3 D_5 - 3D_4 D_3^2 = 0. \tag{35}$$

With (31), (32) (or (33)), (34) all parameters in (25) are determined in terms of the parameters of the QCGLE, so that (25) can be solved for $F(z)$ subject to (35).

3. Traveling Wave Solutions

The nonlinear first order ODEs (9) (with $c = 0$) and (25) can be solved by standard methods yielding $F(z)$ as an inverse function of an elliptic integral, but not $F(z)$ explicitly. Thus, it is obvious to look for another possibility to find elliptic solutions of (9) and (25). With F' according to (9), $c = 0$ necessarily, and hence, taking into account (10)-(12) and (16)-(21), the solution of system (7)-(8) uniquely can be rewritten as

$$\left(F'(z)\right)^2 = \alpha F^4(z) + 4\beta F^3(z) + 6\gamma F^2(z) \tag{36}$$

with

$$\alpha = f, \quad \beta = \frac{b}{4}, \quad \gamma = \frac{a}{6}. \tag{37}$$

Thus, Equation (9) (with $c = 0$) and Equation (36) are equivalent.

Following the same line with F' according to (25) (using (22) or (23), (27)-(35)) we obtain (36), and hence equivalence of (25) and (36). The coefficients in (36) are given by

$$\alpha = \frac{D_4^2}{4b_1^2 D_1^2}, \quad \beta = \frac{\left(ch_1 b_1 + D_2\right) D_4}{8b_1^2 D_1^2}, \quad \gamma = \frac{\left(ch_1 b_1 + D_2\right)^2}{24b_1^2 D_1^2}. \tag{38}$$

The solution of (36) is well known (see [13], Equation (17)). With some algebra we get

$$F(z)=F_0\frac{2\sqrt{\alpha F_0^2+4\beta F_0+6\gamma}\wp'(z)+4\wp^2(z)+(8\gamma+4\beta F_0)\wp(z)-2\gamma\beta F_0-5\gamma^2}{4\wp^2(z)-4\wp(z)(\alpha F_0^2+2\beta F_0+\gamma)+4F_0^2(\beta^2-\alpha\gamma)+4\beta\gamma F_0+\gamma^2}, \tag{39}$$

where $\wp(z;g_2,g_3)$ denotes Weierstrass' elliptic function with invariants $g_2=3\gamma^2, g_3=-\gamma^3$ and $F_0=F(0)$ as an integration constant is the intensity $F(z)$ at $z=0$. The invariants g_2,g_3 can be expressed in terms of the ansatz parameters and the coefficients of the QCGLE. For the case (5) we obtain

$$g_2=\frac{a^2}{12},\quad g_3=-\frac{a^3}{216} \tag{40}$$

with a given by Equation (10). For case (6) we get

$$g_2=3\left(\frac{(ch_1b_1+D_2)^2}{24b_1^2D_1^2}\right)^2,\quad g_3=-\left(\frac{(ch_1b_1+D_2)^2}{24b_1^2D_1^2}\right)^3, \tag{41}$$

with b_1^2 according to Equations (32) or (33) has been used.

Integration of Equations (5) and (6), using Equation (39), yields the phase function $\phi(z)$. For case (5) we obtain

$$\phi(x)=d\cdot\log\left(F_0\frac{2\sqrt{\alpha F_0^2+4\beta F_0+6\gamma}\wp'(x)+4\wp^2(x)+(8\gamma+4\beta F_0)\wp(x)-2\gamma\beta F_0-5\gamma^2}{4\wp^2(x)-4\wp(x)(\alpha F_0^2+2\beta F_0+\gamma)+4F_0^2(\beta^2-\alpha\gamma)+4\beta\gamma F_0+\gamma^2}\right), \tag{42}$$

where $\alpha,\beta,\gamma,g_2,g_3$ given by (37) and (40), respectively and chirp parameter d by (20). Since $c=0$ in this case F and ϕ are stationary. As is known [8], subject to certain conditions, traveling wave solutions can be obtained from the stationary solutions applying a Galilean transformation (see [8], Equation (53))

$$x'=x+vt,\quad t'=t,\quad \psi'(x',t')=\psi(x,t)\mathrm{e}^{i\left(vx+\frac{v^2}{2}t\right)},\quad v=const,$$

to solutions (39) and (42). —We disregard this possibility to find solutions of the QCGLE.

For case (6), integral $\phi(z)=\int(b_0+b_1F(z))\mathrm{d}z$ with $F(z)$ according to (39), cannot be evaluated in closed form (in general). With respect to the example presented below, for particular F_0, integration yields a closed form result. If $F_0=-\frac{\beta}{\alpha},\alpha>0,\beta<0$ (see example in Section 4), solution $F(z)$ reads

$$F(z)=\frac{\frac{\beta^2}{\alpha\sqrt{\alpha}}\wp'(z)-\frac{2\beta}{\alpha}\wp^2(z)-\frac{2\beta^3}{3\alpha^2}\wp(z)+\frac{4\beta^5}{9\alpha^3}}{2\wp^2(z)+\frac{2\beta^2}{3\alpha}\wp(z)-\frac{4\beta^4}{9\alpha^2}} \tag{43}$$

and

$$\phi(z)=\left(b_0-\frac{\beta}{\alpha}b_1\right)z+\frac{b_1}{2\sqrt{\alpha}}\left(\log\left(\wp(z)-\frac{\beta^2}{3\alpha}\right)-\log\left(\wp(z)+\frac{2\beta^2}{3\alpha}\right)\right), \quad (44)$$

where α,β,g_2,g_3 are given by (37) ($\gamma=\frac{2\beta^2}{3\alpha}$), (40) and by

$b_0=\frac{cc_1}{2D_1}, b_1^2=-\frac{D_2D_4}{4D_1D_3}$ with c according to (34).

It should be noted that in both cases (40) and (41) the discriminant of $\wp(z;,g_2,g_3)$ vanishes, so that $\wp(z)$ degenerates to trigonometric ($g_2>0, g_3>0$) or hyperbolic ($g_2>0, g_3<0$) functions, thus depending on the sign of γ in (37) and (38) (see [14], 18.12). Hence, $F(z)$ according to (39) must be rewritten in two versions. In the following, we are preferring to maintain form (39) since it is rather compact and independent on the sign of γ.

Summing up, solutions (1) of Equation (2) can be derived if $\phi'(z)$ and $F(z)$, $z=x-\omega t$, are related by (5) or (6), with $F(z)$ given by (39) subject to certain constraints and conditions for d,c,b_0,b_1,ω. It is necessary to check consistency of the results with the initial assumptions $\{f,\phi,c,\omega,d,b_0,b_1\}\subset\mathbf{R}$. This will be done in the following.

First, we note for case (5) that $\alpha,\beta,\gamma,g_2,g_3$ are real (see Equation (10) with $c=0$). Chirp parameter d is real and hence ω since (20) is satisfied. Constraint (21) is necessary for unique existence of d.

Second, for case (6), $\alpha,\beta,\gamma,g_2,g_3$ are real (see Equations (37), (40)) if c is real according to Equation (34) and hence ω. Thus, real c implies nonnegative radicand in Equation (34). Constraint (35) is necessary for unique b_1^2. —We emphasise that real $\alpha,\beta,\gamma,g_2,g_3$ are important for evaluation of Equations (38), (39).

Third, we note that real g_2,g_3 imply real $\wp(z;,g_2,g_3)$ and $\wp'(z;,g_2,g_3)$ if z is real (see [14], 18.5). Thus, $F(z)$ according to (39), is real, since $\alpha F_0^2+4\beta F_0+6\gamma\ge 0$ due to Equation (38).

Due to the properties of $\wp(z;g_2,g_3)$ (poles and periods) and the dependence of the denominator on α,β,γ,F_0, a singularity analysis of $F(z)$ w.r.t. z on the basis of (39) is very difficult. Indeed, what can be stated is that $F(z)$ exhibits only poles w.r.t. z, consistent with a Theorem by Conte and Ng [7]. To conclude, there is consistency between the results obtained and the initial assumptions for $f,\phi,c,\omega,d,b_0,b_1$.

As is known [15], that Equation (38) alone is suitable to study the nature of the solution $F(z;F_0,\alpha,\beta,\gamma)$ by considering the graphs $\{(F')^2,F\}$ of Equation (38) (denoted as "phase diagram" in the following). Types are depicted in **Figure 1**.

It is clear that the choice of the intensity $F_0>0$ in relation to the zeros of $(F')^2$ ($F_1=0, F_{2,3}=-\frac{2\beta\pm\sqrt{4\beta^2-6\alpha\gamma}}{\alpha}$) is essential for the singularity behaviour of $F(z)$. $F_0>0$ must be chosen such that $(F')^2_{|F=F_0}\ge 0$. In this manner certain

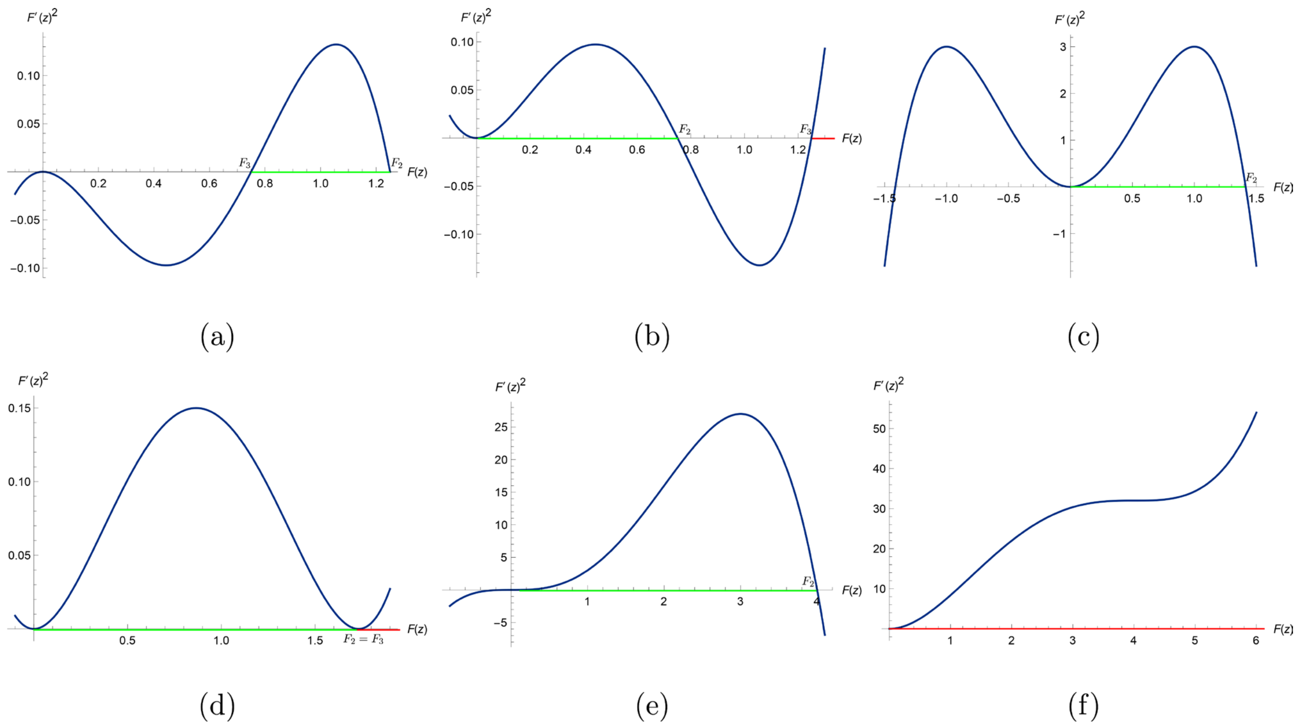


Figure 1. Phase diagrams (types) $\left\{\left(F'(z)\right)^2, F(z)\right\}$ according to Equation (37) corresponding to bounded or singular solutions $F(z)$. Green domains: bounded solutions; red domains: singular solutions. (a) $0<F_3\le F_0\le F_2, \alpha<0, \gamma<0, \beta>0, 3\gamma\alpha<2\beta^2$, periodic $F(z)$. (b) $0<F_0\le F_2, \alpha>0, \gamma>0, \beta<0, 3\gamma\alpha<2\beta^2$, pulse-like $F(z)$; $F_0>F_3$, singular. (c) $0<F_0\le F_2, \alpha<0, \gamma>0, \beta\in\mathbf{R}$, pulse-like $F(z)$. (d) $0<F_0\le F_3=F_2, \alpha>0, \gamma>0, \beta<0, 3\gamma\alpha=2\beta^2$, kink-like $F(z)$; $F_0>F_2=F_3$, singular. (e) $0<F_0\le F_2, \alpha<0, \gamma=0, \beta>0$, algebraic pulse-like $F(z)$. (f) $2\beta^2<3\gamma\alpha, \alpha>0, \gamma>0$, unbounded $F(z)$ for any $F_0>0$. Comments in the text.

domains are defined (labelled green or red in **Figure 1**), where $\left(F'(z)\right)^2$ is bounded (in a finite interval) or unbounded (in an infinite interval), respectively. Characterisation in the phase diagram conditions (PDCs) (a)-(f) in the captions of **Figure 1** (periodic, pulse-like, kink-like) is well known in the literature [15].

The foregoing results can be summarised as follows. Bounded or unbounded solutions $\Re\left(\psi(x,t)\right)=\sqrt{F(x-ct)}\cos\left(\phi(x-ct)-\omega t\right)$ of Equation (2) exist if $\phi'(z)$ and $F(z)$, $z=x-ct$ are related by $\phi'(z)=d\dfrac{F'(z)}{F(z)}$ or $\phi'(z)=b_0+b_1F(z)$, if the amplitude $F(z)$ satisfies the ODE $\left(F'(z)\right)^2=\alpha F^4(z)+4\beta F^3(z)+6\gamma F^2(z)$, and if the parameters of the QCGLE satisfy the PDCs associated to **Figure 1** together with certain constraints (see, e.g., (21), (35)). The parameter range for existence of these particular solutions is the subspace (in parameter space) defined by the PDCs and the constraints.

4. Examples

To elucidate the foregoing results, we first consider $\phi'(z)=d\dfrac{F'(z)}{F(z)}$ with $c=0$.

Needless to say, that if all parameters are prescribed, constraints (21) are not satisfied in general. Due to (21) one of the parameters $\{\varepsilon, h_j, c_j, j=1,3,5\}$ cannot be prescribed. As a solution of (21), this parameter must be inserted into (20) leading to (lengthy) expressions for d, hence for $\omega, \alpha, \beta, \gamma, g_2, g_3$ and, finally, to $F(x)$ and $\phi(x)$ according to (39) and (42), respectively. For simplicity, we assume a particular solution of (21) w.r.t. h_3 and h_5. If

$$h_3 = \frac{2c_3 h_1}{c_1}, \quad h_5 = \frac{3c_5 h_1}{c_1}, \quad sign\frac{c_3}{c_1} = \pm 1, \tag{45}$$

the constraint (21)

$$\frac{-3D_3 \mp \sqrt{9D_3^2 + 8D_2^2}}{4D_2} = \frac{-2D_5 \pm \sqrt{4D_5^2 + 3D_4^2}}{2D_4}$$

(where $\mp$ and $\pm$ corresponds to sign in (44)) is satisfied and c is real (see Equation (34)), we obtain

$$d = -\frac{c_1}{2h_1}, \quad \omega = -\frac{\varepsilon c_1}{h_1} \tag{46}$$

$$\alpha = \frac{4c_5 h_1^2}{c_1 D_1}, \quad \beta = \frac{c_3 h_1^2}{c_1 D_1}, \quad \gamma = -\frac{2\varepsilon h_1}{3D_1} \tag{47}$$

$$g_2 = \frac{4\varepsilon^2 h_1^2}{3D_1^2}, \quad g_3 = \frac{8\varepsilon^3 h_1^3}{27D_1^3}. \tag{48}$$

Subject to (47) the PDC according to **Figures 1(a)-(f)** must be evaluated. For instance, parameters

$$c_1 = -1, \ c_3 = -1, \ c_5 = \frac{1}{8}, \ h_1 = -1, \ h_3 = -2, \ h_5 = \frac{3}{8}, \ \varepsilon = -1, \ F_0 = 4 \tag{49}$$

are consistent with the PDC of **Figure 1(a)**. In this case $F(x)$ (see **Figure 2**) is periodic with period p (see [14], 18.12.30)

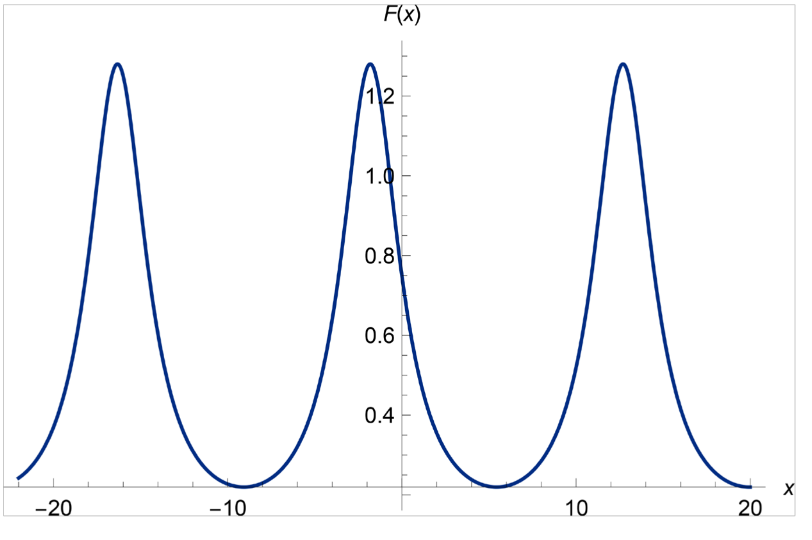


Figure 2. Intensity $F(x)$ according to Equation (37) and parameters (48).

$$p=\frac{\pi\sqrt{D_1}}{\sqrt{\varepsilon h_1}}. \tag{50}$$

A plot of $\Re(\psi(x,t))=\sqrt{F(x)}\cos(\phi(x)-\omega t)$ with $\phi(x)$ according to (42) is shown in **Figure 3**.

The second case $\phi'(z)=b_0+b_1F(z)$ can be exemplified following the line presented before. If

$$c_5=\frac{h_5\left(3c_1c_3^2-4c_3h_1h_3-c_1h_3^2\right)}{c_3^2h_1+4c_1c_3h_3-3h_1h_3^2}, \tag{51}$$

constraint (35) is satisfied. Subject to (51), parameters

$$c_1=-\frac{3}{4},\ c_3=1,\ c_5=-2.5,\ h_1=1,\ h_3=-1,\ h_5=-1,\ \varepsilon=1,\ F_0=\frac{7}{32} \tag{52}$$

are consistent with the PDC of **Figure 1(d)**. Coefficients α,β,γ and invariants g_2,g_3 are given by (37) and (40), respectively. Two field patterns of $\Re(\psi(x,t))$ are shown in **Figure 4(a)** and **Figure 4(b)**. We note that the PDC according to **Figure 1(d)** is the only possible, since α,β,γ according to (39) imply $3\alpha\gamma=2\beta^2$, independent on the choice of $\{\varepsilon,h_j,c_j,j=1,3,5\}$. As mentioned above, the unbounded ("spiky") solution $F(z)$, depicted in **Figure 4(b)**, appears because F_0 is greater than the double root $-\frac{3\gamma}{\beta}(\approx 1.73)$ of $\alpha F^4(z)+4\beta F^3(z)+6\gamma F^2(z)=0$.

As is known [16] [17] [18] phase diagram analysis is an effective approach to study existence and parameter dependence of the solutions $F(z)$ of the nonlinear ODE (36). For instance, parameters (49) correspond to a physical (bounded and nonnegative) solution $F(x)$ according to (39) and hence to a physical $\psi(x,t)$. The associated PDC together with the corresponding constraint(s) define a subspace and thus a range of parameter variation. The (particular) constraint

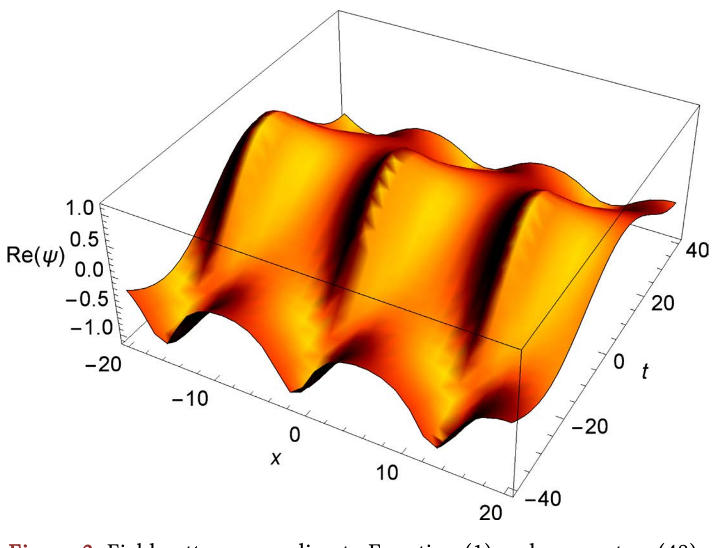
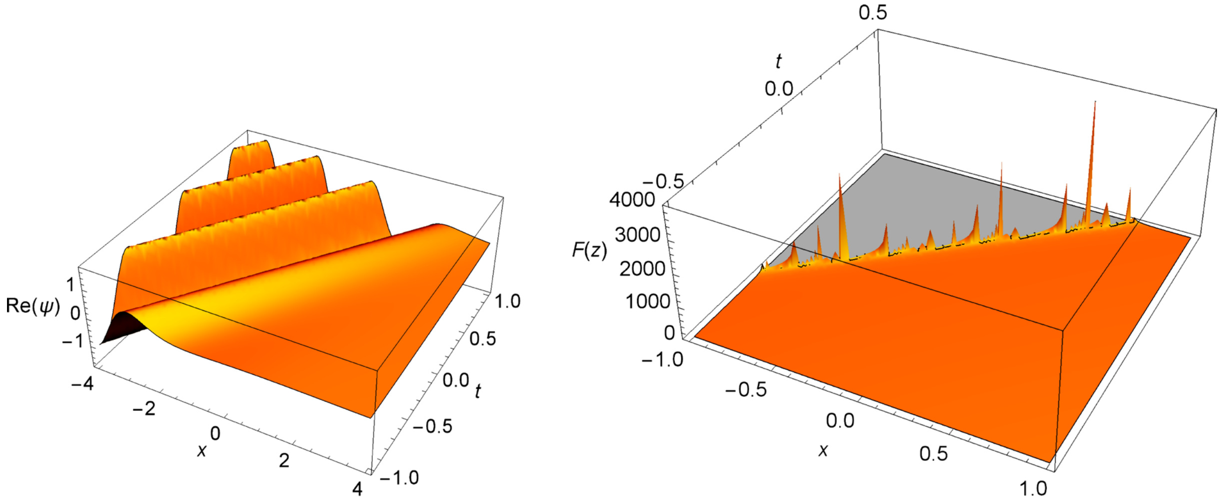


Figure 3. Field pattern according to Equation (1) and parameters (48).

$$\frac{-3D_3-\sqrt{9D_3^2+8D_2^2}}{4D_2}=\frac{-2D_5+\sqrt{4D_5^2+3D_4^2}}{2D_4}$$

is satisfied by (45). Thus the range $\{F_0,\varepsilon,c_1,c_3,c_5,h_1\}$ is defined by the PDC of Figure 1(a) only. With parameters (see (49)) $c_1=-1$, $c_3=-1$, $h_1=-1$, $h_3=\frac{2c_3h_1}{c_1}$, $h_5=\frac{3c_5h_1}{c_1}$, evaluation of the PDC leads to the range $\{F_0,\varepsilon,c_5\}$ depicted in Figure 5. With parameters $c_5=\frac{1}{8}$, $h_3=\frac{2c_3h_1}{c_1}$, $h_5=\frac{3c_5h_1}{c_1}$, $\varepsilon=-1$, $F_0=4$, evaluation of the PDC yields to the range $\{c_1,c_3,h_1\}$ depicted



(a) (b)

Figure 4. Field patterns according to Equation (1) and parameters (51): (a) $F_0=\frac{7}{32}$; (b) $F_0=6$. Comments in the text.

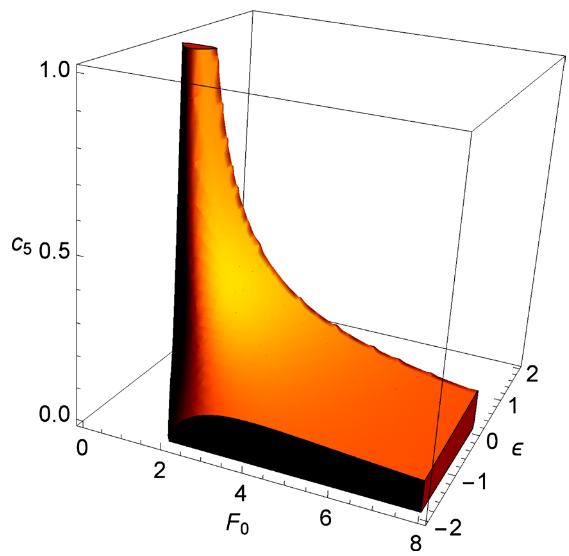


Figure 5. Parameter region for allowed $\{F_0,\varepsilon,c_5\}$ according to phase diagram Figure 1(a) and parameters: $c_1=-1,c_3=-1,h_1=-1,h_3=-2,h_5=0.375$.

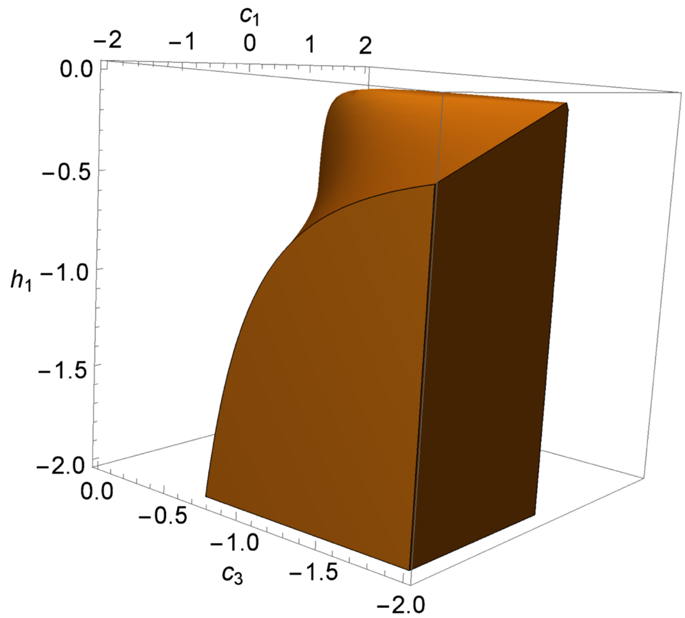


Figure 6. Parameter region for allowed $\{c_1, c_3, h_1\}$ according to phase diagram **Figure 1(a)** and parameters: $c_5 = \frac{1}{8}, h_3 = -2, h_5 = 0.375, \varepsilon = -1, F_0 = 4$.

in **Figure 6**. Further triples of $\{F_0, \varepsilon, c_1, c_3, c_5, h_1\}$ can be considered correspondingly. Thus the whole range of parameter variation of (49), associated to solution $\Re(\psi(x,t))$, depicted in **Figure 3**, can be determined. —Needless to say, this kind of considerations also applies to solutions corresponding to **Figures 1(b)-(e)**.

5. Conclusions

In conclusion, we presented an approach to obtain closed-form traveling wave solutions of the QCGLE. The central assumptions are restrictions on the dependence between $\phi'(z)$ and $F(z)(= f^2(z))$ according to Equations (5) and (6). The solution $F(z)$ is compactly represented by Equation (39) in terms of Weierstrass elliptic function $\wp(z, g_2, g_3)$ (disregarding the fact that $\wp$ is degenerating due to the vanishing discriminant of $\wp$). As a consequence, the phase function $\phi(z)$ can be represented in closed form analytically (see Equations (42) and (44)). The behavior of $F(z)$ is studied by means of a phase diagram approach leading to conditions for "physical" (periodic, pulse-like, kink-like) solutions $F(z)$ as well as to conditions for unbounded ("spiky") solutions $F(z)$. In particular, we obtained the remarkable result that no bounded solution exists if the parameters of the QCGLE satisfy the condition $2\beta^2 - 3\gamma\alpha < 0$, where α, β, γ are given by (37) or (38) and irrespective of $F_0 > 0$. —The phase diagram approach is also suitable to investigate the parameter dependence of solutions (see Section 4).

Finally, we compare our approach with some other methods for getting solutions of the QCGLE.

1) The "simple technique" presented in [8] leads to some results that are consistent with corresponding results above: Assumption (5) in [8] is essentially the same as Equation (5) above. Results (11) and (12) in [8] are the same as (20) and (19) above, respectively. It seems that (14) in [8] is identical with (21) above. Periodic solutions $F(z)$ are not presented in [8], though they are possible solutions of Equation (34) in [8], that has the same structure as Equation (36) above. By using (38) with (10)-(12), it seems that (36) above and (34) in [8] are consistent. Nevertheless, $f(t)$ according to (35) in [8] is not the general solution of Equation (34) in [8].

2) Based on numerical simulations, extreme amplitude solutions of the QCGLE are reported in [19]. Whether they are related to the unbounded solutions above (see **Figure 4(b)**) is an open question.

3) The particular relations (3.38 a,b), (3.51 a,b), (3.57 a,b) used in [2] lead to exact solutions that describe fronts (kinks), pulses, sources and sinks. Obviously, as in [9], periodic solutions are not presented.

4) In [9], based on a Laurent expansion ansatz and a particular relation between amplitude and phase function (see Equations (12), (14) in [9]), exact solutions are derived (see Equation (20) in [9]).

5) In our estimation, even if we take into account recent publications ([3], and references therein), the most comprehensive general (without particular relations between phase and amplitude) treatment to obtain exact solutions of the QCGLE is presented in [4](a) [5] ([11], Equations (33), (34)) ([4](b), Equation (52)) [6] ([7], Equation (50)). Comparing the solutions in [4](b) and [11] with solutions (39), (42) and (43), (44) above, we first note that the number of constraints are different. Solutions (37), (42) are associated to <u>two</u> constraints as one of the four possibilities of (21); the particular solutions (33), (34) in [11] are connected with 3 and 5 constraints, respectively. As mentioned above, the general solution (52) in [4](b) exists subject to 5 constraints. With respect to the formal representation of amplitude and phase, we secondly note that results in [4](b) and [11] are simpler than our results above. The same holds for Equation (50) in [7]. However it seems that—in the general treatment—the question of defining conditions for physical solutions is not addressed. This is not a trivial problem (if, for example, solution (52) in [4](b) is considered, meromorphic M must be non-negative and ϕ' must be real). These requirements must be discussed subject to the (well known) analytical properties of $\wp^2(\xi)$ and $\wp(2\xi)$ and subject to 5 constraints). —Obviously, the phase diagram approach is suitable not only for studying the parameter dependence of solutions outlined above, but also for specifying the parameters such that solutions (39), (42), (43), (44) and hence $\Re(\psi(x,t))$ are physical.

6) For the cubic Ginzburg-Landau equation the non-existence of elliptic traveling wave solutions has been proved [20]. So it is obvious to consider our result above if $c_5 = h_5 = 0$. With $\phi'(z) = d\dfrac{F''(z)}{F(z)}$ we obtained $c = 0$ necessarily,

consistent with the proposition in [20]. According to (20), (21) it follows $d = 0$ and hence $\phi(z) = const$. For case $\phi'(z) = b_0 + b_1 F(z)$, we assumed $b_0 = \frac{cc_1}{2D_1}$ for simplicity, leading to $b_1 = 0$ (see (32), (33)), and $b_0 \to \infty$, in contradiction to bounded $\phi'(z)$.

Summing up, for both cases (5) and (6), our results are consistent with [20]. Without assuming $b_0 = \frac{cc_1}{2D_1}$ the question of consistency is open. —With respect to case (6), we conclude, that the non-existence claim in [20] is not valid in the cubic-quintic case.

Directions in which further investigations can go should be indicated. First, it would be interesting to find more relations than (5) and (6) in order to increase the solution set of the QCGLE. Secondly, it would be important to find generalisations for ansatz (5) as well as for (6) by, for example, including a further parameter. Thirdly, if (5) and (6) are modified by "small corrections", the solutions presented above may be taken as start solutions (different from solutions of the NLSE) for a perturbation approach. Finally, a stability analysis of the solutions found with respect to the parameters of the QCGLE as well as with respect to "small" perturbation of *f* seems possible.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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