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Some Results Regarding φ-maps on G-Cone Metric Spaces with Banach Algebra

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

Aims/ objectives: In our study, we used generalized contraction mapping in G-cone metric space with Banach algebras to establish various fixed point and common fixed point results. Beg [1] defines this space in terms of a few contractive conditions on φ-maps. Our outcomes are a generalization and an extension of several well-known fixed point results.

Keywords: Banach algebras; common fixed point; G-cone metric spaces; generalized Lipschitz conditions.

2020 Mathematics Subject Classification: 47H10, 54H25.

1 Introduction

Fixed point theory is a crucial area of research in mathematics. To establish the existence and uniqueness properties of fixed points, the Banach contraction principle is often employed [2]. This principle requires certain geometric assumptions about the spaces or mapping structures to obtain the fixed point of contractive mappings. Consequently, finding the fixed point of contractive mappings is a common topic of mathematical inquiry.

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Dhage [3] first proposed the idea of D-metric space in his work. Most of the findings asserted by Dhage and others are legitimate, albeit, as later research has shown that some theorems involving this space were incorrect. These mistakes were noted by Mustafa and Sims [4], who also suggested a new structure that generalizes metric spaces known as G-metric spaces.

In a recent study, Huang and Zhang [5] extended the definition of metric spaces to include ordered Banach spaces beyond the real number set. This led to the development of cone metric spaces, in which they introduced the concept of completeness and defined sequences of convergence. They also demonstrated some fixed point theorem of contractive mapping on a complete cone metric space, assuming a normal cone. Following their work, other authors have studied the fixed point theorem for both normal and non-normal cones.

In their recent research paper, Beg, Abbas, and Nazir [1] introduced the concept of generalized cone metric spaces, which are an extension of both generalized metric spaces and cone metric spaces. They demonstrated various topological properties such as convergence and completeness of these spaces. Additionally, they obtained a few fixed-point results satisfying contractive conditions. Pari and Vetro [6] proved some theorems related to φ-maps in cone metric spaces. Similarly, W. Shatanawi [7] obtained a few fixed-point results in generalized metric spaces.

In a later paper, Adewale and Osawaru [8] presented the idea of a generalized cone metric space, which could be a more common space than a generalized metric space and cone metric space. To prove the properties of convergence of sequences and some fixed point results in this area, they replaced the set of real numbers with an ordered Banach space.

The reason for this work is to get a few fixed point and common fixed point results of φ-maps that fulfill contractive conditions in G-cone metric spaces with Banach algebra. Our outcomes are generalizations of a few results found in existing literature.

2 Preliminaries

Assume that $\mathcal{A}$ is a Banach algebra. $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined following properties(for all $u, v, w \in \mathcal{A}, \alpha \in \mathbb{R}$):

(i) $(uv)w = u(vw)$;

(ii) $u(v + w) = uv + uw$ & $(u + v)w = uw + vw$;

(iii) $\alpha(uv) = (\alpha u)v = u(\alpha v)$;

(iv) $||uv|| \leq ||u||.||v||$;

Now assume that a Banach algebra has a unit (multiplicative identity) e such that $eu = ue = u$ for all $u \in \mathcal{A}$. An element $u \in \mathcal{A}$ is called invertible if there exists $u^{-1} \in \mathcal{A}$ such that

$$uu^{-1} = u^{-1}u = e.$$

for more details, we refer to [9].

The following proposition is well known (see [9]).

Proposition 2.1. *[10][9] Consider $u \in \mathcal{A}$ be a Banach algebra with a unit e, then spectral radius $\rho(u)$ of $u \in \mathcal{A}$ holds*

$$\rho(u) = \lim_{n\to\infty} ||u_n||^{\frac{1}{n}} = inf||u^n||^{\frac{1}{n}} < 1$$

further, $(e - u)$ is inverse and

$$(e - u)^{-1} = \sum_{i=0}^{\infty} u^i$$

.

Let θ be the null vector, e be the identity element of $\mathcal{A}$, and a subset $\mathcal{P}$ of $\mathcal{A}$ is called a cone if satisfies the following:

(i) $\{\theta, e\} \subset \mathcal{P}$ & $\mathcal{P}$ to be closed;

(ii) $\mathcal{P}^2 = \mathcal{P}\mathcal{P} \subset \mathcal{P}$;

(iii) $\alpha\mathcal{P} + \beta\mathcal{P} \subset \mathcal{P}$, $\forall$ non-negative real no. α & β;

(iv) $\mathcal{P} \cap (-C\mathcal{P}) = \{\theta\}$.

With cone $\mathcal{P}$, a partial ordering $\preceq$ is defined as $u \preceq w$ iff $(w-u) \in \mathcal{P}$ & $u \prec w$ if $u \preceq w$ & $u \neq w$ whereas $u \ll w$ means $(w-u) \in int\mathcal{P}$.

If Banach space $\mathcal{A}$ & cone $\mathcal{P}$ satisfies the conditions 1, 3 and 4 then $\mathcal{P}$ [9] is called a cone of $\mathcal{X}$.

Definition 2.1. [11][10] Let $\mathcal{X}$ be a non-empty set, $\mathcal{A}$ be a Banach algebra & $\mathcal{P} \subseteq \mathcal{A}$ be a cone. Suppose that the mapping $d : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ satisfies the conditons $\forall u, v, w \in \mathcal{X}$:

(i) $d(u, w) = \theta$ iff $u = w$, & $\theta \preceq d(u, w)$;

(ii) $d(u, w) = d(w, u)$;

(iii) $d(u, w) \preceq d(u, v) + d(v, w)$ for all $u, v, w \in \mathcal{X}$.

d is called a cone metric and $(\mathcal{X}, d)$ is called cone metric space with a Banach algebra $\mathcal{A}$. Here $d(u, v) \in \mathcal{P}$ for every $u, v \in \mathcal{X}$.

Definition 2.2. [8][12] Let $\mathcal{X}$ be a non-empty set, $\mathcal{A}$ be a Banach algebra & $G : \mathcal{X}^3 \to \mathcal{A}$ be a function satisfying the properties:

(i) $G(u, v, w) = \theta$ iff $u = v = w$;

(ii) $\theta \prec G(u, v, w), \forall u, v \in X,$ with $u \neq v$;

(iii) $G(u, u, v) \preceq G(u, v, w), \forall u, v, w \in X,$ with $w \neq v$;

(iv) $G(u, v, w) = G(v, w, u) = G(u, w, v) =$ ... (symmetry);

(v) $G(u, v, w) \preceq G(u, a, a) + G(a, v, w), \forall u, v, w \in \mathcal{X}$ (rectangle inequality).

A G-cone metric over the Banach algebra $\mathcal{A}$ is referred to as G, and the G-cone metric space with the Banach algebra is denoted by $(\mathcal{X}, G)$.

Definition 2.3. [12][10] Assume $(\mathcal{X}, G)$ be a cone metric space with Banach algebra $\mathcal{A}$ and $\{u_n\}$ a sequence in $\mathcal{X}$.Then we say

(i) $\{u_n\}$ is a convergent sequence if, for all $c \in A$ with $\theta \ll c,$ and an N such that $d(u_n, u) \ll c$ for all $n \geq N.$ We write it by $u_n \to u (n \to \infty).$

(ii) $\{u_n\}$ is a Cauchy sequence if, for all $c \in A$ with $\theta \ll c,$ and an N such that $d(u_n, u_m) \ll c$ for all $n, m \geq N.$

(iii) $(\mathcal{X}, d)$ is a complete cone metric space if all Cauchy sequences in $\mathcal{X}$ are convergent.

Lemma 2.1. *[11][10] Let $\mathcal{A}$ be a Banach algebra and m, a vector in $\mathcal{A}$. If $0 \leq r(m) < 1$, then*

$$r((e-m)^{-1}) < (1 - r(m))^{-1}$$

.

Lemma 2.2. *[13][10] Let $\mathcal{A}$ be a Banach algebra and u, v be vectors in $\mathcal{A}$. If u and v commute, then the following holds:*

(i) $r(uv) \leq r(u)r(v)$;

(ii) $r(u+v) \leq r(u)+r(v)$;

(iii) $|r(u)-r(v)| \leq r(u-v)$.

Lemma 2.3. *[13][10] If $\mathcal{A}$ is real Banach algebra with a solid cone $\mathcal{P}$ and $\{u_n\}$ is a sequence in $\mathcal{A}$. Suppose $||u_n|| \to 0 (n \to \infty)$ for any $\theta \ll c$. Then $u_n \ll c$ for all $n > N^1, N^1 \in N$.*

Lemma 2.4. *[8][10] If E is a real Banach space with a solid cone $\mathcal{P}$ and if $||u_n|| \to 0$ as $n \to \infty$, then for any $\theta \ll c$, there exists $N \in \mathbb{N}$ such that, for any $n > N$, we have $u_n \ll c$.*

Lemma 2.5. *[14][10] Let $\mathcal{A}$ be a Banach algebra and $k \in A$. If $\rho(k) < 1$, then*

$$\lim_{n\to\infty} ||k^n|| = 0$$

.

Definition 2.4. [8][12] Let $(\mathcal{X}, G)$ be a G-cone metric space over Banach algebra. G is said to be symmetric if

$$G(u,v,v) = G(u,u,v)$$

for all $u, v, w \in \mathcal{X}$.

Lemma 2.6. *[14][10] Let $\mathcal{X}$ be a symmetric G-cone metric space, then*

$$d_G(u,v) = 2G(u,v,v).$$

Example 2.7. *[15][10] Let $\mathcal{A}$ be the Banach space of functions $C(M)$ on a compact Hausdorff topological space M, with multiplication. Then $\mathcal{A}$ is a Banach algebra, and the constant function $f(t) = 1$ is the unit of $\mathcal{A}$. Let $P = \{f \in \mathcal{A} : f(t) \geq 0$ for every $t \in M\}$. Then $\mathcal{P} \subset \mathcal{A}$ is a normal cone with a normal constant M = 1. Let $\mathcal{X} = C(M)$ with the metric $d : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ defined by*

$$d(f,g) = |f(t) - g(t)|$$

where $t \in M$. Then $(\mathcal{X}, d)$ is a cone metric space over a Banach algebra $\mathcal{A}$.

Example 2.8. *[12] Let $(\mathcal{X}, d)$ be a cone metric space. Define $G : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{A}$, by*

$$G(u,v,w) = d(u,v) + d(v,w) + d(u,w).$$

Definition 2.5. [12] Let $(\mathcal{X}, G)$ be a G-cone metric space over Banach algebra $\mathcal{A}$ is said to be G-bounded if for any $u, v, w \in \mathcal{X}$, there exists $M \succ \theta$ such that

$$||G(u,v,w)|| \preceq M$$

Definition 2.6. [12] Let $(\mathcal{X}, G)$ be a G-cone metric space over Banach algebra and $\{u_n\}$ a sequence in $\mathcal{X}, c \gg \theta$ with $c \in \mathcal{A}$. Then

(i) $\{u_n\}$ converges to $u \in \mathcal{X}$ iff $G(u_m, u_n, u) \ll c$ for all $n, m > N^1, N^1 \in N$.

(ii) $\{u_n\}$ is Cauchy sequence if and only if $G(u_n, u_m, u_p) \ll c$ for every $n, m > p > N^1, N^1 \in N$.

(iii) $(\mathcal{X}, G)$ is complete G-cone metric space with Banach algebra if every Cauchy sequence converges.

Definition 2.7. [16] Suppose f and g are maps of a set $\mathcal{X}$. If $w = fu = gu$ for some u in $\mathcal{X}$, then u is called a coincidence point of f and g, and w is called a coincidence point of f and g.

Lemma 2.9. *[17] Suppose that f and g are weakly compatible maps of set $\mathcal{X}$. If f and g have a unique coincidence point $w = fu = gu$, then w is the unique common fixed point of f and g.*

Definition 2.8. [17] Maps $f, g : \mathcal{X} \to \mathcal{X}$ are weakly compatible, if for all $u \in \mathcal{X}$, then

$$fgu = gfu$$

whenever $gu = fu$.

Definition 2.9. [6] Let $\mathcal{P}$ be the above-defined cone. Function of nondecreasing $\varphi : \mathcal{P} \to \mathcal{P}$ is called a φ-map if conditions hold,

(i) $\varphi(\theta) = \theta$ and $\theta < \varphi(w) < w$ for $w \in \mathcal{P} \backslash \{\theta\}$,

(ii) $w \in int\mathcal{P}$ implies $w - \varphi(w) \in int\mathcal{P}$,

(iii) $\lim_{n\to\infty} \varphi^n(w) = \theta$ for every $w \in \mathcal{P} \backslash \{\theta\}$.

3 Main Result

Theorem 3.1. *Let $(\mathcal{X}, d)$ be a complete symmetric G-cone metric space with a Banach algebra $\mathcal{A}$ and $\mathcal{P}$ be a nonnormal cone. Assume that the maps $f, g : \mathcal{X} \to \mathcal{X}$ satisfies the contractive condition*

$$G(fu, fv, fw) \preceq \varphi G(gu, gv, gw) \tag{3.1}$$

for all $u, v, w \in \mathcal{X}$. Suppose f and g are weakly compatible with $f(\mathcal{X}) \subset g(\mathcal{X})$. If $f(\mathcal{X})$ or $g(\mathcal{X})$ is a complete subspace of $\mathcal{X}$, then maps f and g have a unique common fixed point.

Proof. Let u_0 be an arbitrary element in the set $\mathcal{X}$. We can then choose another element $u_1 \in \mathcal{X}$ such that $fu_0 = gu_1$. This is possible because $f(\mathcal{X})$ is a subset of $g(\mathcal{X})$. We can continue this process indefinitely, such that having chosen $u_n \in \mathcal{X}$, we can choose $u_{n+1} \in \mathcal{X}$ such that $fu_n = gu_{n+1}$ for all $n \in N$.

If, at some point, we have $fu_n = fu_{n-1}$ for some $n \in N$, then we know that $fu_m = fu_n$ for all $m \in N$ with $m > n$. Thus, the sequence $\{fu_n\}$ is a Cauchy sequence.

Assuming that $fu_n \neq fu_{n-1}$ for all $n \in N$, by (3.1), we have

$$\begin{aligned} G(fu_{n+1}, fu_{n+1}, fu_n) &\preceq \varphi G(gu_{n+1}, gu_{n+1}, gu_n) \\ &= \varphi G(fu_n, fu_n, fu_{n-1}) \\ &\preceq \varphi^2 G(gu_n, gu_n, gu_{n-1}) \\ &= \varphi^2 G(fu_{n-1}, fu_{n-1}, fu_{n-2}) \\ &\preceq ....\varphi^n G(fu_1, fu_1, fu_0) \end{aligned}$$

Given $\theta \ll l$ and select a positive real number η such that $\{ l - \varphi(l) + N(\theta + \eta)\} \subset int\mathcal{P}$, where $N(\theta + \eta) = \{u \in \mathcal{A} : \|u\| < \eta\}$. Also, select a natural number N such that $\varphi^n G(fx_1, fu_1, fu_0) \ll l - \varphi(l)$ for all $m \geq N$. Consequently

$$G(fu_{m+1}, fu_{m+1}, fu_m) \ll l - \varphi(l)$$

$\forall m \geq N$ and $m \geq N$ we have

$$G(fu_m, fu_{n+1}, fu_{n+1}) \ll l \tag{3.2}$$

$\forall n \geq m$. We assume that (3.2) holds for some $n \geq m$, and then show that it also holds when n = m. By using definition(2.4)

$$\begin{aligned} G(fu_m, fu_{n+2}, fu_{n+2}) &\preceq G(fu_m, fu_{m+1}, fu_{m+1}) + G(fu_{m+1}, fu_{n+2}, fu_{n+2}) \\ &\ll l - \varphi(l) + \varphi G(gu_{m+1}, gu_{n+2}, gu_{n+2}) \\ &\ll l - \varphi(l) + \varphi G(fu_m, fu_{n+1}, fu_{n+1}) \\ &\ll l - \varphi(l) + \varphi(l) \\ &= l \end{aligned}$$

We can prove that (3.2) holds for all $m, n \geq N$ by induction, using the fact that it holds when $m = n + 1$. This implies that fu_n is a Cauchy sequence. Suppose $f(\mathcal{X})$ is a complete subspace of $\mathcal{X}$. Then, there exists $w \in f(\mathcal{X}) \subset g(\mathcal{X})$ such that $fu_n \to w$ and $gu_n \to w$. Let v be an element of $\mathcal{X}$ such that $gv = w$. We now need to show that $gv = fv$.

Let us fix $\theta \ll l$. Next we can choose a natural number N that satisfies condition $G(w, fu_n, fu_n) \ll \frac{l}{2}$ and $G(gu_n, gv, gv) \ll \frac{l}{2}$. Then by using the definition(2.4),

$$\begin{aligned} G(w, fv, fv) &\preceq G(w, fu_n, fu_n) + G(fu_n, fv, fv) \\ &\preceq G(w, fu_n, fu_n) + \varphi G(gu_n, gv, gv) \end{aligned}$$

By using the definition of φ we get

$$\begin{aligned} F(G(w, fv, fv)) &< [G(w, fu_n, fu_n) + G(gu_n, gv, gv)] \\ &\ll [\frac{l}{2} + \frac{l}{2}] \\ &= l \end{aligned}$$

We are given that $G(w, fv, fv) \ll \frac{l}{i}$ holds for all $i \geq 1$. Also, we have $\frac{l}{i} - G(w, fv, fv) \in \mathcal{P}$ for all i, and as i approaches infinity, we get $-G(w, fv, fv) \in \mathcal{P}$. Since $G(w, fv, fv) \in \mathcal{P}$, we can conclude that $G(w, fv, fv) = \theta$ which implies that $gv = fv = w$. Therefore, w is a coincidence point of f and g.

To show that w is a common fixed point of f and g, we need to use the concept of weak compatibility of the maps. As $fv = gv$, it follows from the weak compatibility of f and g that w is indeed a common fixed point of f and g.

$$fw = fgv = gfv = gw.$$

We demonstrate that $fw = gw = w$. If $gw \neq w$, we get from condition (3.1):

$$\begin{aligned} G(fw, fw, fv) &\preceq \varphi G(gw, gw, gv) \\ &< G(gw, gw, gv) \\ &= G(fw, fw, fv) \end{aligned}$$

Let us start by assuming that f and g are mappings such that $fw = w = gw$, implying that w is a common fixed point for both f and g.

Now, let w^* be another common fixed point of f and g. To prove this, we will make use of equation (3.1).

$$\begin{aligned} G(w, w, w^*) &= G(fw, fw, fw^*) \\ &\preceq \varphi G(gw, gw, gw^*) \\ &= G(gw, gw, gw^*) \\ &< G(w, w, w^*) \end{aligned}$$

To obtain uniqueness, we need to resolve the contradiction. □

Theorem 3.2. *Let $\mathcal{X}$ be a symmetric G-cone metric space that is complete and based on a Banach algebra $\mathcal{A}$. Suppose $\mathcal{P}$ is a non-normal cone in $\mathcal{A}$ and let $f : \mathcal{X} \to \mathcal{X}$ be a mapping that satisfies a contractive condition.*

$$G(fu, fv, fw) \preceq \varphi N(u, v, w) \tag{3.3}$$

where

$$N(u, v, w) \in \{G(u, v, w), G(u, fu, fu), G(v, fv, fv), G(fu, v, w)\} \tag{3.4}$$

$\forall u, v, w \in \mathcal{X}$, then map f has a unique fixed point in $\mathcal{X}$.

Proof. Choose $u_0 \in \mathcal{X}$. Let $u_n = fu_{n+1}$ for all $n \in N$. Suppose that $u_n \neq u_{n-1}$, for each $n \in N$. Thus we have

$$G(u_n, u_{n+1}, u_{n+1}) = G(fu_{n-1}, fu_n, fu_n) \preceq \varphi N(u_{n-1}, u_n, u_n)$$

where

$$\begin{aligned} N(u_{n-1}, u_n, u_n) \in \{&G(u_{n-1}, u_n, u_n), G(u_{n-1}, fu_{n-1}, fu_{n-1}), \\ &G(u_n, fu_n, fu_n), G(fu_{n-1}, u_n, u_n)\} \\ = \{&G(u_{n-1}, u_n, u_n), G(u_{n-1}, u_n, u_n), \\ &G(u_n, u_{n+1}, u_{n+1}), G(u_n, u_n, u_n)\} \\ = \{&G(u_{n-1}, u_n, u_n), G(u_n, u_{n+1}, u_{n+1}), \theta\} \end{aligned}$$

If $N(u_{n-1}, u_n, u_n) = G(u_n, u_{n+1}, u_{n+1})$,then

$$G(u_n, u_{n+1}, u_{n+1}) \preceq \varphi G(u_n, u_{n+1}, u_{n+1})$$

By using the definition of φ maps, we have

$$G(u_n, u_{n+1}, u_{n+1}) < G(u_n, u_{n+1}, u_{n+1})$$

That is impossible. If $N(u_{n-1}, u_n, u_n) = \theta$, then

$$G(u_n, u_{n+1}, u_{n+1}) \preceq \varphi(\theta) < \theta$$

That is a contradiction. And at last, if

$$N(u_{n-1}, u_n, u_n) = G(u_{n-1}, u_n, u_n),$$

Then

$$G(u_n, u_{n+1}, u_{n+1}) \preceq \varphi G(u_{n-1}, u_n, u_n)$$

By applying a similar method to Theorem 3.1, we can conclude that the sequence $\{u_n\}$ is a Cauchy sequence. Since $\mathcal{X}$ is complete, $\{u_n\}$ converges to a limit $t \in \mathcal{X}$. We now need to show that $t = t$. For any $n \in \mathbb{N}$, we can use the definition 2.4 to show that this is true.

$$\begin{aligned} G(t, t, ft) &\preceq G(t, t, u_n) + G(u_n, u_n, ft) \\ &= G(t, t, u_n) + G(fu_{n-1}, fu_{n-1}, ft) \\ &\preceq G(t, t, u_n) + \varphi N(u_{n-1}, u_{n-1}, t) \end{aligned}$$

And

$$\begin{aligned} N(u_{n-1}, u_{n-1}, t) \in \{&G(u_{n-1}, u_{n-1}, t), G(u_{n-1}, fu_{n-1}, fu_{n-1}), \\ &G(u_{n-1}, fu_{n-1}, fu_{n-1}), G(fu_{n-1}, u_{n-1}, t)\} \\ = \{&G(u_{n-1}, u_{n-1}, t), G(u_{n-1}, u_n, u_n), G(u_n, u_{n-1}, t). \end{aligned}$$

Choose a natural number N_1 such that $G(t,t,u_n) \ll \frac{l}{2}$, for all $n \geq N_1$. We investigate these situations as follows;
Case I: If $N(u_{n-1},u_{n-1},t) = G(u_{n-1},u_{n-1},t)$, then

$$\begin{aligned} G(t,t,ft) &\preceq G(t,t,u_n) + \varphi G(u_{n-1},u_{n-1},t) \\ &< G(t,t,u_n) + G(u_{n-1},u_{n-1},t) \\ &\ll \frac{l}{2} + \frac{l}{2} \\ &= l \end{aligned}$$

Case II: If $N(u_{n-1},u_{n-1},t) = G(u_{n-1},u_n,u_n)$, then

$$\begin{aligned} G(t,t,ft) &\preceq G(t,t,u_n) + \varphi G(u_{n-1},u_n,u_n) \\ &< G(t,t,u_n) + G(u_{n-1},u_n,u_n) \\ &\ll \frac{l}{2} + \frac{l}{2} \\ &= l \end{aligned}$$

Case III: If $N(u_{n-1},u_{n-1},t) = G(u_n,u_{n-1},t)$, then

$$\begin{aligned} G(t,t,ft) &\preceq G(t,t,x_n) + \varphi G(u_n,u_{n-1},t) \\ &< G(t,t,u_n) + G(u_n,u_{n-1},t) \\ &\preceq G(t,t,u_n) + G(u_n,u_{n-1},u_{n-1}) + G(u_{n-1},u_{n-1},t) \\ &\ll l \end{aligned}$$

For any natural number n, this formula applies universally. $G(t,t,ft) \ll \frac{l}{i}, \forall i \geq 1$. So $\frac{l}{i} - G(t,t,ft) \in \mathcal{P}$,for all $i \geq 1$. Since $\frac{l}{i} \to 0$ as $i \to \infty$ and $\mathcal{P}$ is closed, hence $-G(t,t,ft) \in \mathcal{P}$ and $G(t,t,ft) = \theta$ therefore $t = ft$. □

Theorem 3.3. *Let $\mathcal{X}$ be a complete G-cone metric space over a Banach algebra $\mathcal{A}$ and let $\mathcal{P}$ be a non-normal cone in $\mathcal{A}$. Suppose that f is a self-map of $\mathcal{X}$ which satisfies the following condition for all u, v, and w in $\mathcal{X}$:*

$$G(fu,fv,fw) \preceq \mu N(u,v,w) \tag{3.5}$$

where

$$N(u,v,w) \in \{G(u,v,w), G(u,fu,fu), G(v,fv,fv), G(w,fw,fw), \\ \frac{[G(u,fv,fv) + G(w,fu,fu)]}{2}, \frac{G(u,Tv,Tv) + G(v,Tu,Tu)}{2}\}$$

Assume f is a function from $\mathbb{R}$ to $\mathbb{R}$, and μ is a constant satisfying $0 \leq \mu < 1$. Then f has a unique fixed point.

Proof. By taking $\varphi(u) = \mu(u)$ with $\mu \in [0,1)$ and using the same method as in Theorem 3.2, we can obtain the desired result. □

Theorem 3.4. *Consider a complete G-cone metric space $\mathcal{X}$ over a Banach algebra $\mathcal{A}$ and a non-normal cone $\mathcal{P}$ in $\mathcal{A}$. Let f be a self-map of $\mathcal{X}$ that satisfies the following condition for all $u,v,w \in \mathcal{X}$:*

$$G(fu,fv,fw) \preceq \mu N(u,v,w) \tag{3.6}$$

where

$$N(u,v,w) \in \{G(u,v,w), G(u,fv,fw), G(v,fv,fv), \\ G(u,u,fv), G(v,v,fu), G(w,w,fw)\}$$

or

$$N'(u,v,w) \in \{G(u,v,w), G(u,u,fu), G(v,v,fv), \\ G(u,u,fv), G(v,v,fw), G(w,w,fw)\}$$

and μ *is a constant satisfying* $0 \leq \mu < 1$. *Then we say that* f *has a unique fixed point.*

Proof. Suppose that f satisfies (3.6). By using (3.6) with $w = v$ we have

$$G(fu,fv,fv) \preceq \mu N(u,v,v) \tag{3.7}$$

we have

$$N(u,v,v) \in \{G(u,v,v), G(u,fu,fu), G(v,fv,fv), \\ G(u,u,fv), G(v,v,fu)\}$$

And

$$N'(u,v,v) \in \{G(u,v,v), G(u,u,fu), G(v,v,fv), \\ G(u,u,fv), G(v,v,fu)\}$$

By Lemma 2.13 we know that $d_G(u,v) = 2G(u,v,v)$ makes $\mathcal{X}$ to cone metric space

$$N(u,v) \in \{d_G(u,v), d_G(u,fv), d_G(v,fv), \\ d_G(u,fv), d_G(v,fu)\}$$

Let $u_0 \in X$ and $u_n = fu_{n-1}$. Suppose that $u_n \neq u_{n+1}$, then

$$d_G(u_n,u_{n+1}) = d_G(fu_{n-1},fu_n) \preceq \mu N''(u_{n-1},u_n)$$

where

$$N''(u_{n-1},u_n) \in \{d_G(u_{n-1},u_n), d_G(u_n,u_{n+1}), d_G(u_{n-1},u_{n+1}), \theta\}$$

We investigate some possibilities:

Case I: If $N''(u_{n-1},u_n) = d_G(u_{n-1},u_{n+1})$, then

$$\begin{aligned} d_G(u_n,u_{n+1}) &\preceq \mu d_G(u_{n-1},u_{n+1}) \\ &\preceq \mu\{d_G(u_{n-1},u_n) + \mu d_G(u_n,u_{n+1})\} \\ &\preceq \frac{\mu}{e-\mu} d_G(u_{n-1},u_n) \end{aligned}$$

Case II: If $N''(u_{n-1},u_n) = d_G(u_{n-1},u_{n+1})$, then

$$d_G(u_n,u_{n+1}) \preceq \mu d_G(u_n,u_{n+1})$$

we have $d_G(u_n,u_{n+1})(1-\mu) \preceq \theta$, since $\mu \in [0,1)$ this a contradiction.

Case III: If $N''(u_{n-1},u_n) = \theta$, then

$$d_G(u_n,u_{n+1}) \preceq \mu\theta$$

which contradict with the assumption of $u_n \neq u_{n+1}$

Case IV: If $N''(u_{n-1}, x_n) = d_G(u_{n-1}, u_n)$, then

$$\begin{aligned} d_G(u_n, u_{n+1}) &\preceq \mu d_G(u_{n-1}, u_n) \\ &\preceq \mu^2 d_G(u_{n-2}, u_{n-1} \\ &\preceq .... \preceq \mu^n d_G(u_0, u_1) \end{aligned}$$

By the method used in [18] Theorem 2.3, we can obtain the desired result of f having a unique fixed point. □

Example 3.5. *Let $\mathcal{A} = R^3$ and $\mathcal{P} = \{u \in R : u \geq 0\}$ be a cone. Let $X = [1, \infty)$ with the following G-cone metric space*

$$G(u, v, w) = d(u, v) + d(v, w) + d(u, w)$$

and the usual metric $d(u, v) = |u - v|$. Define the two maps $f, g : \mathcal{X} \to \mathcal{X}$ by

$$fu = u,$$

$$gu = 4u - 3,$$

for all $u \in \mathcal{X}$. Let's define the function $\varphi : \mathcal{P} \to \mathcal{P}$ by $\varphi q = \frac{1}{3}q,$ for all $q \in \mathcal{P}$. Then we have

$$\begin{aligned} G(fu, fv, fw) &= d(fu, fv) + d(fv, fw) + d(fu, fw) \\ &= |fu - fv| + |fv - fw| + |fu - fv| \\ &= |u - v| + |v - w| + |u - w| \\ &\preceq \frac{4}{3}(|u - v| + |v - w| + |u - w|) \\ &= \frac{1}{3}(|4u - 4v| + |4v - 4w| + |4u - 4w|) \\ &= \frac{1}{3}(|4u - 3 - 4v + 3| + |4v - 3 - 4w + 3| + |4u - 3 - 4w + 3|) \\ &= \frac{1}{3}(|gu - gv| + |gv - gw| + |gu - gw|) \\ &= \frac{1}{3}G(gu, gv, gw) \\ &\preceq \varphi G(gu, gv, gw) \end{aligned}$$

$$f1 = g1 = 1$$

Consequently, we used the conditions of results 3.1 and see that $u = 1$ could be a unique common fixed point for f and g.

4 Conclusion

We have generalized existing fixed point and common fixed point results for φ-map in G-cone metric spaces with Banach algebra $\mathcal{A}$.

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Competing Interests

The authors have stated that they do not have any conflicts of interest to declare.

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