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The Existence of Meromorphic Solutions to Non-Linear Delay Differential Equations

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Abstract

In this paper, we study the existence of the transcendental meromorphic solution of the delay differential equations

$$w(z+1)w(z-1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^k=R(z,w(z))=\frac{P(z,w(z))}{Q(z,w(z))},\quad \text{where } a(z)$$

is a rational function, $P(z,w(z))$ and $Q(z,w(z))$ are polynomials in $w(z)$ with rational coefficients, k is a positive integer. Under the assumption when above equations own transcendental meromorphic solutions with minimal hyper-type, we derive the concrete conditions on the degree of the right side of them. Specially, when $w(z)=0$ is a root of $Q(z,w(z))$, its multiplicity is at most k. Some examples are given here to illustrate that our results are accurate.

Keywords

Non-Linear Delay Differential Equations, Painlevé Type Equations, Nevanlinna Theory, Meromorphic Function Solutions, Minimal Hypertype

1. Introduction

In actual life, a completely linear system does not exist due to part of the system itself have varying degrees of non-linear properties and the influence of external conditions. At the same time, through the industrial production process and in natural social sciences, there are many practical systems like the well-known network control transmission system, water and power system, communication system, urban traffic management system, etc., which are related to the state of a certain moment in the past, and the characteristics of the system are called delay. We can see that it is very necessary to study nonlinear time-delay system, among which, nonlinear delay differential equation is a vital tool for studying nonlinear time-delay system, we study nonlinear delay differential equation to characterize a part of the corresponding nonlinear time-delay system, so as to obtain the characteristics of nonlinear time-delay system and solve some practical problems.

The differential Painlevé equations over the complex domain and the Painlevé type equations are a special and significant class of nonlinear delay differential equations with important applications in physics. In 2000, Ablowitz *et al.* [1] applied Nevanlinna theory in difference equations of complex domains, studied the following equations:

$$w(z+1)+w(z-1)=R(z,w(z))$$

$$w(z+1)w(z-1)=R(z,w(z))$$

and obtained some results on the degree of the right side of the equations.

Subsequently, some well-known theories and approaches which are widely used in the study of differential and difference equations have emerged such as the difference version of the logarithmic derivative lemma (Halburd, Korhonen [2] and Chiang, Feng [3]) and so on ([4] [5] [6]).

In the year of 2007, Halburd and Korhonen [7] discovered a discrete version of the Painlevé III and obtained the following theorem:

Theorem 1.1. Let $w(z)$ be an admissible finite-order meromorphic solution of the equation

$$w(z+1)w(z-1)=\frac{c_2\left(w(z)-c_+\right)\left(w(z)-c_-\right)}{\left(w(z)-a_+\right)\left(w(z)-a_-\right)}=R(z,w),$$

where the coefficients are meromorphic functions, $c_2 \not\equiv 0$ and $\deg_w(R)=2$. If the order of the poles of $w(z)$ is bounded, then either $w(z)$ satisfies a difference Riccati equation

$$w(z+1)=\frac{pw(z)+q}{w(z)+s}$$

where $p,q,s \in S(w)$, ($S(w)$ is a set of small functions of $w(z)$), or equation can be transformed by a bilinear change in $w(z)$ to one of the equations

$$w(z+1)w(z-1)=\frac{\gamma(z-1)w^2(z)+\delta(z)\lambda^z w(z)+\gamma(z)\mu(z)\lambda^{2z}}{\left(w(z)-1\right)\left(w(z)-\gamma(z)\right)},$$

$$w(z+1)w(z-1)=\frac{w^2(z)+\delta(z)\mathrm{e}^{\frac{\pi z i}{2}}}{w^2(z)-1}$$

where $\lambda \in C$ and $\delta,\mu,\gamma \in S(w)$ are arbitrary finite-order periodic functions such that δ and γ have period 2 and μ has period 1.

In 2017, Halburd and Korhonen [8] applied Nevanlinna theory ([9] [10]) to consider the existence of the meromorphic solutions of complex differential-difference equations with the hyper-order less than 1 and obtain the following theorem.

Theorem 1.2. Let $w(z)$ be a non-rational meromorphic solution of

$$w(z+1)-w(z-1)+a(z)\frac{w'(z)}{w(z)}=R(z,w(z))=\frac{P(z,w(z))}{Q(z,w(z))},$$

where $a(z)$ is rational, $P(z,w(z))$ is a polynomial in $w(z)$ having rational coefficients in *z*, and $Q(z,w)$ is a polynomial in $w(z)$ with roots that are non-zero rational functions of *z* and not roots of $P(z,w(z))$. If the hyper-order of $w(z)$ is less than one, then

$$\deg_w(P)=\deg_w(Q)+1\le 3 \text{ or } \deg_w(R)\le 1.$$

Inspiring by the above work, Liu and Song [11] contemplated the non-linear difference equations

$$w(z+1)w(z-1)+a(z)\frac{w'(z)}{w(z)}=R(z,w(z))=\frac{P(z,w(z))}{Q(z,w(z))}.$$

We can't help but considering how the result would be different when $\frac{w'(z)}{w(z)}$ in the above equation turns to a more general $\left(\frac{w'(z)}{w(z)}\right)^k$. That's the main purpose of this paper. Here are the main contents and conclusions of this article:

Theorem 1.3. Suppose that *k* is a positive integer and that $a(z)$ is a rational function. Let $w(z)$ be a transcendental meromorphic solution of

$$w(z+1)w(z-1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^k=R(z,w(z))=\frac{P(z,w(z))}{Q(z,w(z))}, \tag{1.1}$$

where $P(z,w)$ and $Q(z,w)$ are two coprime polynomials in $w(z)$ having rational coefficients in *z*. If $\limsup_{r\to\infty}\frac{\log^+ T(r,w)}{r}=0$, then

$$\deg_w(R(z,w))\le 2k+2$$

and one of the following holds for $w(z)=0$ is not a root of $Q(z,w)$.

(i) when $\deg_w(Q(z,w))=0$ or $\deg_w(Q(z,w))=1$, we have

$$\deg_w(P(z,w))\le k+1;$$

(ii) when $\deg_w(Q(z,w))\ge 2$, we have

$$\deg_w(Q(z,w))<\deg_w(P(z,w))\le k+\deg_w(Q(z,w)).$$

In particular, when $w=0$ is a root of $Q(z,w)$, its multiplicity is at most *k*.

In fact, a meromorphic function which satisfies $\limsup_{r\to\infty}\frac{\log^+ T(r,w)}{r}=0$ is called minimal-hypertype where $T(r,w(z))$ is Nevanlinna characteristics of $w(z)$. Now we list some examples below to demonstrate that our results are accurate.

Example 1.1. The meromorphic function $w(z)=\csc \pi z$ with

$$\limsup_{r\to\infty}\frac{\log^{+}T(r,w)}{r}=0$$

solves

$$w(z+1)w(z-1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^{2}=(a(z)+1)w^{2}(z)-a(z)$$

We can obtain that $\deg_{w}(P(z,w))=2<k+1=3$ since $k=2$.

Example 1.2. The meromorphic function $w(z)=\frac{1}{\mathrm{e}^{z}+1}$ solves

$$w(z+1)w(z-1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^{2}=\frac{P(z,w(z))}{Q(z,w(z))},$$

$$\begin{aligned}P(z,w(z))=&\left[2\left(e+\frac{1}{e}-2\right)+5\left(e+\frac{1}{e}\right)+14\right]w^{4}(z)-\left[5\left(e+\frac{1}{e}\right)-14\right]a(z)w^{3}(z)\\&+\left[1-7a(z)-3\left(e+\frac{1}{e}a(z)\right)\right]w^{2}(z)+a(z)\left(e+\frac{1}{e}-4\right)w(z)+a(z)\end{aligned}$$

and $Q(z,w(z))=\left[2-e-\frac{1}{e}\right]w^{2}(z)+\left[\frac{1}{e}+e-2\right]w(z)+1$, where $a(z)(\not\equiv 0)$ is an arbitrary rational function. Obviously, we have

$$\limsup_{r\to\infty}\frac{\log^{+}T(r,w)}{r}=0$$

and

$$\deg_{w}(P(z,w))=4\le\deg_{w}(Q)+2.$$

This paper is organized as follows. In Section 2, we outline the lemma we need to use. The main results discussed on different situations are summarized in Section 3.

2. Auxiliary Lemmas

We present some lemmas which play important role in the following. The first one is the difference version of the logarithmic derivative lemma for meromorphic functions with minimal hyper-type due to Zheng and Korhonen [12].

Lemma 2.1. [[12], Theorem 1.2] Let $w(z)$ be a meromorphic function. If

$$\limsup_{r\to\infty}\frac{\log^{+}T(r,w)}{r}=0,$$

then

$$m\left(r,\frac{w(z+c)}{w(z)}\right)=o(T(r,w)),$$

holds for a constant c as $r(\notin E)\to\infty$, where E is a subset of $[1,+\infty)$ with the zero upper density, that is

$$\overline{dens}E=\limsup_{r\to\infty}\frac{1}{r}\int_{E\cap[1,r]}\mathrm{d}t=0.$$

Lemma 2.2. [[12], Lemma 2.1] Let $G(r)$ be a nondecreasing positive function in $[1,+\infty)$ and logarithmic convex with $G(r)\to+\infty(r\to+\infty)$. Assume that

$$\limsup_{r\to+\infty}\frac{\log G(r)}{r}=0.$$

Set

$$\phi(r)=\max_{1\le t\le r}\left\{\frac{t}{\max\{1,\log G(t)\}}\right\}.$$

Then given a real number $\delta\in\left(0,\frac{1}{2}\right)$, one has

$$G(r)\le G\left(r+\phi^{\delta}(r)\right)\le\left(1+4\phi^{\delta-\frac{1}{2}}(r)\right)G(r),r\notin E_{\delta},$$

where E_{δ} is a subset of $[1,+\infty)$ with the zero upper density.

Lemma 2.3. [[13], Lemma 19] Let $w(z)$ be a non-rational meromorphic solution of

$$P(z,w)=0,$$

where $P(z,w)$ is a differential-difference polynomial in $w(z)$ with rational coefficients, and let $a_1,a_2,\cdots,a_k$ be rational functions satisfying $P(z,a_j)\not\equiv 0$ for all $j\in\{1,2,\cdots,k\}$.

If there exists $s>0$ and $\tau\in(0,1)$ such that

$$\sum_{j=1}^{k}n\left(r,\frac{1}{w-a_j}\right)\le k\tau n(r+s,w)+O(1),$$

then $\limsup\limits_{r\to\infty}\frac{\log^{+}T(r,w)}{r}>0$.

If the right side of (1.1) is a polynomial in $w(z)$, we can obtain the following fact.

Lemma 2.4. Let $w(z)$ be a non-rational meromorphic solution of the equation

$$w(z+1)w(z-1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^{k}=P(z,w(z)),\left(k\in N^{+}\right),\tag{2.1}$$

where k is a positive integer, where $a(z)$ is a rational function and $P(z,w(z))$ is a polynomial of $w(z)$ on z. If $\limsup\limits_{r\to\infty}\frac{\log^{+}T(r,w)}{r}>0$, then $\deg_w\left(P(z,w(z))\right)\le k+1$.

The proof of Lemma 2.4. Assume that $\deg_w\left(P(z,w(z))\right)=p\ge k+2$. Firstly, we consider that $w(z)$ have finitely many poles and zeros, it's obvious that there exist a rational function $f(z)$ and an entire function $g(z)$ such that

$$w(z)=f(z)\mathrm{e}^{g(z)}.$$

On the basis of Equation (2.1), we will get

$$e^{2g(z)} f(z+1) f(z-1) \frac{e^{g(z+1)+g(z-1)}}{e^{2g(z)}} + a(z)\left(\frac{f'(z)}{f(z)} + g'(z)\right)^k = P\left(z, f(z)e^{g(z)}\right). \quad (2.2)$$

Using the Lemma 2.1, one knows

$$T\left(r, e^{g(z+1)-g(z)}\right) = m\left(r, e^{g(z+1)-g(z)}\right) = S\left(r, e^{g(z)}\right),$$

$$T\left(r, e^{g(z-1)-g(z)}\right) = m\left(r, e^{g(z-1)-g(z)}\right) = S\left(r, e^{g(z)}\right).$$

At this point and the fact that $e^{g(z)}$ is transcendental, it follows from (2.2) that

$$\deg_w\left(P(z, w(z))\right) T\left(r, e^{g(z)}\right) \le 2T\left(r, e^{g(z)}\right) + S\left(r, e^{g(z)}\right).$$

This is a contradiction since $\deg_w\left(P(z, w(z))\right) = p \ge k+2$ for $k \in N^+$.

In the following, we consider that either $w(z)$ has infinitely many zeros or $w(z)$ has infinitely many poles (or both). Since the coefficients of (2.1) are rational, we can always choose a zero or a pole $z = z_j$ of $w(z)$ in such a way that there is no cancellation with the coefficients. At this time, we continue to discuss the following two different situations:

Case 1. $z = z_j$ is a pole of $w(z)$ with multiplicity l, then $w(z+1)w(z-1)$ has a pole with multiplicity $pl(>k)$ at $z = z_j$.

Subcase 1.1. $z_j + 1$ is a pole of $w(z)$ with multiplicity t ($1 \le t \le pl$) and $z_j - 1$ is a pole of $z_j - 1$ with multiplicity $pl - t$. By shifting (2.1) forward and backward, we have

$$w(z+2)w(z) + a(z+1)\left(\frac{w'(z+1)}{w(z+1)}\right)^k = P\left(z+1, w(z+1)\right),$$

$$w(z)w(z-2) + a(z-1)\left(\frac{w'(z-1)}{w(z-1)}\right)^k = P\left(z-1, w(z-1)\right).$$

Analyzing the poles on both sides of the above equations and we will know $z_j + 2$ is a pole of $w(z)$ with multiplicity $pt - l$, $z_j - 2$ is a pole of $w(z)$ with multiplicity $p^2 l - pt - l$. Continue iterating over the equation, one knows $z_j + 3$ is a pole of $w(z)$ with multiplicity $p^2 t - pl - t$, $z_j - 3$ is a pole of $w(z)$ with multiplicity $p^3 l - p^2 t - 2pl + t$, $z_j + 4$ is a pole of $w(z)$ with multiplicity $p^3 t - p^2 l - 2pt + l$, $z_j - 4$ is a pole of $w(z)$ with multiplicity $p^4 l - p^3 t - 3p^2 l + 2pt + l$, and so on. Hence,

$$n\left(d + |z_j|, w\right) \ge p^d l \ge p^d$$

for all $d \in N$. It follows that

$$\limsup_{r\to\infty} \frac{\log^+ T(r,w)}{r} \ge \limsup_{r\to\infty} \frac{\log n(r,w)}{r} \ge \limsup_{d\to\infty} \frac{\log n\left(d + |z_j|, w\right)}{d + |z_j|}$$

$$\ge \limsup_{d\to\infty} \frac{\log p^d}{d + |z_j|} \ge \log(k+2) > 0.$$

This contradicts to $\limsup_{r\to\infty}\frac{\log^+ T(r,w)}{r}=0$, so the assumption is not valid.

Subcase 1.2. z_j+1 is a pole of $w(z)$ with multiplicity $t(t>pl)$, z_j-1 is a zero of $w(z)$ with multiplicity $t-pl$. By shifting (2.1) up, one can deduce that z_j+2 is a pole of $w(z)$ with multiplicity $pt-l$, z_j+3 is a pole of $w(z)$ with multiplicity $p^2t-pl-t$, z_j+4 is a pole of $w(z)$ with multiplicity $p^3t-p^2l-2pt+l$, and so on. Thus,

$$n\left(d+\left|z_j\right|,w\right)\geq p^d l\geq p^d,$$

for all $d\in N$, and so

$$\limsup_{r\to\infty}\frac{\log^+ T(r,w)}{r}\geq\limsup_{d\to\infty}\frac{\log n\left(d+\left|z_j\right|,w\right)}{d+\left|z_j\right|}>0.$$

This contradicts to $\limsup_{r\to\infty}\frac{\log^+ T(r,w)}{r}=0$.

Subcase 1.3. z_j-1 is a pole of $w(z)$ with multiplicity $t(t>pl)$, z_j+1 is a zero of $w(z)$ with $t-pl$, as the proof process of Subcases 1.2, we can push out the contradicts so that the hypothesis $\deg_w\left(P(z,w(z))\right)=p\geq k+2$ is not valid.

Case 2. $w(z)$ has a zero in $z=z_j$ with multiplicity *q*, then $w(z+1)w(z-1)$ has *k*-th poles at $z=z_j$. This implies that at least one of z_j+1 and z_j-1 is a pole of $w(z)$. In the same discussion as in case 1, we can also derive that

$$\limsup_{r\to\infty}\frac{\log^+ T(r,w)}{r}\geq\frac{\log p^{d-1}}{d+\left|z_j\right|}>0.$$

This contradicts to $\limsup_{r\to\infty}\frac{\log^+ T(r,w)}{r}=0$. The Lemma 2.4 is proved.

Finally, we consider the case in which $Q(z,w(z))$ of (1.1) has a non-zero repeated root as a polynomial in $w(z)$.

Lemma 2.5. Suppose that $k,m(\geq 2)$ are two positive integers. Let $w(z)$ be a non-rational meromorphic solution of

$$w(z+1)w(z-1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^k=\frac{P(z,w(z))}{\left(w(z)-b_1(z)\right)^m\hat{Q}(z,w(z))},\tag{2.3}$$

where $a(z)$, $b_1(z)(\not\equiv 0)$ are rational functions, and $\hat{Q}(z,w(z))$ is a polynomial in $w(z)$ such that $\hat{Q}(z,w)\left(w(z)-b_1(z)\right)^m=Q(z,w)$. If $\limsup_{r\to\infty}\frac{\log^+ T(r,w)}{r}=0$, then

$$m+\deg_w\left(\hat{Q}(z,w)\right)<\deg_w\left(P(z,w)\right)<k+m+\deg_w\left(\hat{Q}(z,w)\right).$$

On basis of Lemma 2.5, one can deduce $\limsup_{r\to\infty}\frac{\log^+ T(r,w)}{r}>0$ provided that $k=1$. Next we give the details of the proof of Lemma 2.5 below.

The proof of Lemma 2.5. We can transform (2.3) into $\Phi(z,w(z))=0$, where $\Phi(z,w(z))$ is a differential difference polynomial. Notice that $\Phi(z,b_1(z))\not\equiv 0$, so the first condition of Lemma 2.3 is satisfied.

Suppose that z_j is a zero of $w(z)-b_1(z)$ with multiplicity l, and that neither $b_1(z)$ nor any of the coefficients in (2.3) have a zero or pole at $z=z_j$. Furthermore, if the coefficient functions in (2.3) don't have a zero or a pole at z_j and $z_j+i(i\in Z)$, then z_j is called a *generic zeros*. Since the coefficients of (2.3) are rational, for the case of non-generic zeros, we can know that for integrated counting functions, it will bring an error term $O(\log r)$ at most. Hence, we only need to consider the *generic zeros* in the following.

Case 1. Assume that

$$\deg_w(P(z,w))\le m+\deg_w(\hat{Q}(z,w)).$$

Let $z=z_j$ be a *generic zero* of $w(z)-b_1(z)$ of order l, it follows from (2.3) that $z=z_j$ is a pole of $w(z+1)w(z-1)$ with multiplicity $h(\ge ml>l)$. This implies that at least one of z_j+1 and z_j-1 is a pole of $w(z)$.

- Assume z_j+1 is a pole of $w(z)$ with multiplicity $r(r<h)$ and z_j-1 is a pole of $w(z)$ with multiplicity $h-r$. By shifting (2.3) forward and backward once time, one can deduce that $w(z)$ has a pole of order *k* at z_j+2 and a pole of order *k* at z_j-2. By continuing the iteration, it follows that $w(z)$ has either a finite value or a pole at z_j+3 (or z_j-3). Consequently,

$$\begin{aligned} n\left(r,\frac{1}{w-b_1}\right) &\le \frac{l}{h+2k}n(r+2,w)+O(1) \\ &\le \frac{1}{m}n(r+2,w)+O(1), \end{aligned}$$

since $h\ge ml>l$. Due to the Lemma 2.3 we have $\limsup_{r\to\infty}\frac{\log^+ T(r,w)}{r}>0$, it contradicts the assumption.

- Assume one of z_j+1 and z_j-1 is a pole of $w(z)$ with multiplicity $r(r\ge h)$.

Iterating (2.3) as before we will know that z_j is a pole of $w(z+2)$ with multiplicity *k* or z_j is a pole of $w(z-2)$ with multiplicity *k*, so

$$\begin{aligned} n\left(r,\frac{1}{w-b_1}\right) &\le \frac{l}{r+k}n(r+2,w)+O(1) \\ &\le \frac{1}{m}n(r+2,w)+O(1), \end{aligned}$$

since $r\ge h$ and $h\ge ml$. By Lemma 2.3, we also have $\limsup_{r\to\infty}\frac{\log^+ T(r,w)}{r}>0$. This is a contradiction, so

$$\deg_w(P(z,w))>m+\deg_w(\hat{Q}(z,w)).$$

Case 2. Assume that

$$\deg_w\left(P(z,w)\right) \geq k+m+\deg_w\left(\hat{Q}(z,w)\right).$$

We also let $z=z_j$ be a *generic zero* of $w(z)-b_1(z)$ of order l, it's easy to see that $z=z_j$ is a pole of $w(z+1)w(z-1)$ with multiplicity $h(\geq ml>l)$. It indicates that at least one of z_j+1 and z_j-1 is a pole of $w(z)$. By this time, we also expand into several subcases:

- Assume z_j+1 is a pole of $w(z)$ with multiplicity $r(r<h)$ and z_j-1 is a pole of $w(z)$ with multiplicity $h-r$. Shifting (2.3) forward and backward and we can deduce that $w(z)$ has a pole of order $(p-q)r$ at z_j+2 and a pole of order $(p-q)(h-r)$ at z_j-2. Thus, we have

$$\begin{aligned} n\left(r,\frac{1}{w-b_1}\right) &\leq \frac{l}{h+(p-q)h} n(r+2,w)+O(1) \\ &\leq \frac{1}{m} n(r+2,w)+O(1), \end{aligned}$$

since $h\geq ml>l$. Combining with the Lemma 2.3 and we will get

$$\limsup_{r\to\infty} \frac{\log^+ T(r,w)}{r} > 0$$

which is a contradiction.

- Assume one of z_j+1 and z_j-1 is a pole of $w(z)$ with multiplicity $r(r\geq h)$. By iterating (2.3) as before, we have that z_j is a pole of $w(z+2)$ with multiplicity $(p-q)r$ or z_j is a pole of $w(z-2)$ with multiplicity $(p-q)r$, so

$$\begin{aligned} n\left(r,\frac{1}{w-b_1}\right) &\leq \frac{l}{r+(p-q)r} n(r+2,w)+O(1) \\ &\leq \frac{1}{m} n(r+2,w)+O(1), \end{aligned}$$

since $r\geq h$ and $h\geq ml$. Following from the Lemma 2.3 and we will obtain $\limsup_{r\to\infty} \frac{\log^+ T(r,w)}{r} > 0$. This is a contradiction, so

$$\deg_w\left(P(z,w)\right) < k+m+\deg_w\left(\hat{Q}(z,w)\right).$$

In conclusion, we have proved the Lemma 2.5.

3. The Proof of Theorem 1.3

For the equation (1.1), we proceed to prove that $\deg_w\left(R(z,w)\right)\leq 2k+2$. Taking the Nevanlinna characteristic function of both sides of (1.1), we have

$$\begin{aligned} &T\left(r, w(z+1)w(z-1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^k\right) \\ &= T\left(r, R(z,w(z))\right) = \deg_w(R)T(r,w)+O(\log r) \end{aligned}$$

since the coefficients of $R(z,w)$ and $a(z)$ are rational functions. Furthermore, in view of Lemma 2.2 and the lemma on the logarithmic derivative,

$$\begin{aligned}
&\deg_w\left(R(z,w)\right)T(r,w)\\
&\le T\left(r,w(z+1)\right)+T\left(r,w(z-1)\right)+m\left(r,\left(\frac{w'(z)}{w(z)}\right)^k\right)+N\left(r,\left(\frac{w'(z)}{w(z)}\right)^k\right)+S(r,w)\quad(3.1)\\
&\le 2T(r,w)+kN\left(r,\frac{w'(z)}{w(z)}\right)+S(r,w).
\end{aligned}$$

We can see that $\frac{w'(z)}{w(z)}$ has a pole in $z=z_j$ if and only if $w(z)$ has a pole or zero in $z=z_j$, so

$$N\left(r,\frac{w'(z)}{w(z)}\right)\le \bar{N}\left(r,w(z)\right)+\bar{N}\left(r,\frac{1}{w(z)}\right). \tag{3.2}$$

Together with (3.1) and (3.2), we have

$$\deg_w\left(R(z,w)\right)T(r,w)\le (k+2)T(r,w)+S(r,w),$$

and thus $\deg_w\left(R(z,w)\right)\le 2k+2$.

Next, let us complete the proof of the Theorem 1.3. If $\deg_w\left(Q(z,w)\right)=0$, it follows from Lemma 2.4 that $\deg_w\left(P(z,w)\right)\le k+1$. When

$$1\le \deg_w\left(Q(z,w)\right)\le 2k+2,$$

we consider the following two cases.

Case 1. $w(z)=0$ is not a root of $Q(z,w(z))$.

Subcase 1.1. Assume that $\deg_w(Q)=1$, without loss of generality, we set $Q(z,w(z))=w(z)-b_1(z)$, where $b_1(z)\ne 0$ is a rational function. Thus (1.1) can be rewritten as

$$w(z+1)w(z-1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^k=\frac{P(z,w(z))}{w(z)-b_1(z)}. \tag{3.3}$$

It is easy to see that $b_1(z)$ is not a solution of (3.3), so the first condition of Lemma 2.3 is satisfied. Suppose that

$$\deg_w\left(P(z,w)\right)\ge k+2\left(k\in N^+\right)$$

and that z_j is a *generic zero* of $w(z)-b_1(z)$ with multiplicity l. It follows that $w(z+1)w(z-1)$ has a pole at $z=z_j$ of order at least l.

If z_j+1 is a pole of $w(z)$ with multiplicity $r(0<r<l)$ and z_j-1 is a pole of $w(z)$ with multiplicity $l-r$. By Shifting (3.3) up, we have

$$w(z+2)w(z)+a(z+1)\left(\frac{w'(z+1)}{w(z+1)}\right)^k=\frac{P(z+1,w(z+1))}{w(z+1)-b_1(z+1)}, \tag{3.4}$$

and thus $w(z)$ has a pole of order $(k+1)r$ at $z=z_j+2$. Similarly, from (3.3) one can obtain that

$$w(z)w(z-2)+a(z-1)\left(\frac{w'(z-1)}{w(z-1)}\right)^k=\frac{P(z-1,w(z-1))}{w(z-1)-b_1(z-1)} \tag{3.5}$$

and that $w(z)$ has a pole of order $(k+1)(l-r)$ at $z=z_j-2$. Therefore,

$$n\left(r,\frac{1}{w-b_1}\right)\le\frac{l}{l+(k+1)l}n(r+2,w)+O(1)$$
$$\le\frac{1}{3}n(r+2,w)+O(1).$$

Combining with the Lemma 2.3 and we will get $\limsup_{r\to\infty}\frac{\log^+T(r,w)}{r}>0$, which contradicts to the fact in Theorem 1.3.

If z_j+1 or z_j-1 is a pole of $w(z)$ with multiplicity $r(r\ge h)$, it follows from (3.4) and (3.5) that z_j is a pole of $w(z+2)$ with multiplicity $(k+1)r$ or z_j is a pole of $w(z-2)$ with multiplicity $(k+1)r$, then we can also obtain

$$n\left(r,\frac{1}{w-b_1}\right)\le\frac{1}{3}n(r+2,w)+O(1),$$

which is impossible, so we have

$$\deg_w\left(P(z,w)\right)\le k+1,$$

thus, in view of this fact and Lemma 2.4, the first result (*i*) of Theorem 1.3 is proved.

Subcase 1.2. Assume that $2\le\deg_w Q(z,w(z))\le 2k+2$. If $Q(z,w(z))$ of (1.1) has at least a non-zero repeated root as a polynomial in $w(z)$, it follows from Lemma 2.5 that

$$\deg_w\left(Q(z,w)\right)<\deg_w\left(P(z,w)\right)<k+\deg_w\left(Q(z,w)\right).$$

Set $\deg_w\left(Q(z,w)\right)=q$. Now we consider the case of all non-zero roots of $Q(z,w(z))$ are simple, say $b_1(z),\cdots,b_q(z)$, then (1.1) can be written as

$$w(z+1)w(z-1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^k=\frac{P(z,w(z))}{\left(w(z)-b_1(z)\right)\cdots\left(w(z)-b_q(z)\right)\hat{Q}(z)},\quad(3.6)$$

where $\hat{Q}(z)(\not\equiv 0)$ is a polynomial in *z*. It is obviously that $b_1(z),b_2(z),\cdots,b_q(z)$ are not solutions of the above equation, so it satisfies the first condition of Lemma 2.3. The aim is to prove the inequality

$$\deg_w\left(Q(z,w)\right)<\deg_w\left(P(z,w)\right)\le k+\deg_w\left(Q(z,w)\right).$$

Let $z=z_j^i\,(i=1,2,\cdots,q)$ be *generic zero* of $w(z)-b_i(z)$ with multiplicity l_i. If $\deg_w\left(P(z,w)\right)\le\deg_w\left(Q(z,w)\right)$, considering the zeros of $w(z)-b_i(z)$ of (3.6), for example $w-b_1(z)$, we know that $w(z+1)w(z-1)$ has a pole of order at least l_1 at $z=z_j^1$.

Assume z_j^1+1 is a pole of $w(z)$ with multiplicity $r(0<r<l)$ and z_j^1-1 is a pole of $w(z)$ with multiplicity $l-r$. Shifting (3.6) forward and backward and we can obtain that $w(z)$ has a pole of order *k* at z_j^1+2 and a pole of order *k* at z_j^1-2. Thus, we have

$$n\left(r,\frac{1}{w-b_1}\right)\le\frac{l}{l+2k}n(r+2,w)+O(1).\quad(3.7)$$

Assume z_j^1+1 or z_j^1-1 is a pole of $w(z)$ with multiplicity $r(r\geq l)$. By iterating (3.6) as before, we have that z_j is a pole of $w(z+2)$ with multiplicity *k* or z_j is a pole of $w(z-2)$ with multiplicity *k*, we also have

$$n\left(r,\frac{1}{w-b_1}\right)\leq\frac{l}{r+k}n(r+2,w)+O(1). \tag{3.8}$$

Similarly, we can also obtain (3.7) and (3.8) for any $i=2,3,\cdots,q$. So, we get $\limsup_{r\to\infty}\frac{\log^+T(r,w)}{r}>0$ through the Lemma 2.3, which is a contradiction. Consequently, we obtain $\deg_w(P(z,w))>\deg_w(Q(z,w))$.

Next, we turn to the proof of another side of the inequality, that is

$$\deg_w(P(z,w))\leq\deg_w(Q(z,w))+k\leq\min\left\{\deg_w(P(z,w))+k,2k+2\right\}.$$

Assume that $\deg_w(P(z,w))>\deg_w(Q(z,w))+k$ and let z_j^1 be a *generic zero* of $w(z)-b_1(z)$ with multiplicity l_1. From (3.6), we know that $w(z+1)w(z-1)$ has a pole of order at least l_1 in $z=z_j^1$. Here we only consider the case of z_j^1+1 is a pole of $w(z)$ with multiplicity at least l_1. By shifting (3.6) up,

$$\begin{aligned}&w(z+2)w(z)+a(z+1)\left(\frac{w'(z+1)}{w(z+1)}\right)^k\\&=\frac{P(z+1,w(z+1))}{(w(z+1)-b_1(z+1))\cdots(w(z+1)-b_q(z+1))\hat{Q}(z+1)}.\end{aligned}$$

It follows that $z=z_j^1+2$ is a pole of $w(z)$ with multiplicity at least kl_1, and thus

$$n\left(r,\frac{1}{w-b_1}\right)\leq\frac{l_1}{l_1+kl_1}n(r+2,w)+O(1)\leq\frac{1}{2}n(r+2,w)+O(1),$$

since $k\in N^+$. Lemma 2.3 indicates that $\limsup_{r\to\infty}\frac{\log^+T(r,w)}{r}>0$, which contradicts the assumption. Therefore, the result

$$\deg_w(Q(z,w))<\deg_w(P(z,w))\leq k+\deg_w(Q(z,w))$$

is proved.

Case 2. $w(z)=0$ is a root of $Q(z,w(z))$. We shall prove that $w(z)=0$ is a zero of $Q(z,w)$ with the multiplicity at most *k*. Suppose that $Q(z,w)=w^m(z)\tilde{Q}(z,w)$, $m\geq k+1$ and $\tilde{Q}(z,w)$ is a polynomial in $w(z)$ with degree at most $k+1$. Then (1.1) can be rewritten as

$$w(z+1)w(z-1)+a(z)\left(\frac{w'(z)}{w(z)}\right)^k=\frac{P(z,w(z))}{w^m(z)\tilde{Q}(z,w(z))}. \tag{3.9}$$

Let z_j be a *generic zero* of $w(z)$ with multiplicity l. Then z_j is a pole of $w(z+1)w(z-1)$ with multiplicity at lease ml since $m\geq k+1$. Without loss of generality, we only consider the case when $w(z+1)$ has a pole of order ml at $z=z_j$. By shifting (3.9) up, one has

$$w(z+2)w(z)+a(z+1)\left(\frac{w'(z+1)}{w(z+1)}\right)^k=\frac{P(z+1,w(z+1))}{w^m(z+1)\tilde{Q}(z+1,w(z+1))}. \quad (3.10)$$

If $\deg_w(P(z,w))\le m+\deg_w(\tilde{Q}(z,w))$, it follows from the above equation that z_j is a pole of $w(z+2)$ with multiplicity $l+k$, and thus $w(z_j+3)$ could be finite. This means that

$$n\left(r,\frac{1}{w}\right)\le\frac{l}{ml+k+l}n(r+2,w)+O(1),$$

where $\frac{l}{ml+k+l}<\frac{1}{m+1}<\frac{1}{3}$. Notice that $w(z)=0$ is not a solution of (3.9). Using Lemma 2.3, we can obtain $\limsup_{r\to\infty}\frac{\log^+T(r,w)}{r}>0$, which is a contradiction.

If $\deg_w(P(z,w))=\deg_w(Q(z,w))+i$ for $1\le i\le k$, it follows from (3.10) that z_j is a pole of $w(z+2)$ with multiplicity $iml+l$. Hence,

$$n\left(r,\frac{1}{w}\right)\le\frac{1}{(i+1)m+1}n(r+2,w)+O(1),$$

where $\frac{1}{(i+1)m+1}<\frac{1}{3}$. On the basis of Lemma 2.3 we can also get a contradiction. This completes the proof of Theorem 1.3.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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