



# Linear Hybrid Multistep Block Method for Direct Solution of Initial Value Problems of Third Order Ordinary Differential Equations

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## Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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## Abstract

In this article, we focus on linear hybrid multistep method for direct solution of initial value problems of third order ordinary differential equations without reduction to system of first-order ordinary differential equations. The derivation of the method involved using collocation and interpolation techniques with power series as basis function to produce a system of linear equations. The unknown parameters in the system of equations were obtained through the Gaussian elimination technique. The values of the determined parameters were then substituted and evaluated at different grid and off-grid points to produce the required continuous block method. The discrete scheme obtained from the method is self-starting with improved accuracy and a larger interval of absolute stability. Basic properties of the method were investigated. The results showed that the method is zero stable, consistent, convergent and of order seven. The performance of the method was tested by solving linear and nonlinear problems of general third order ordinary differential equations. The result were found to compare favourably with some existing methods in literature.

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## 1 Introduction

This article considers the numerical solution of initial value problems (ivps) of general third order ordinary differential equations is of the form:

$$y''' = f(x, y, y', y''), y(0) = y_0, y' = y'_0, y'' = y''_0. \tag{1.1}$$

In the past, third order ordinary differential equations (1) were traditionally solved by reducing them to first order initial value problem of ordinary differential equations and then solving the resulting equations using any of the available numerical methods. Method In this category include [1] which was designed to serve stiff system of first order of initial value problem of ordinary differential equations. These numerical techniques and its attendant drawbacks had been extensively referenced by many authors, See [2]-[5]. As a result of this, alternative solutions for these problems had been considered by several authors. The following are the existing methods for solving higher-order differential equations without reduction to a set of first order ordinary differential equations [6]-[13] but just to mention the few while [14] developed a hybrid numerical method with block extension for direct solution of initial value problems of third-order ordinary differential equations consider method of the form:

$$y_{\frac{j}{2}} = \sum_{j=1}^{k-1} \alpha_{\frac{j}{2}}(x)y_{n+\frac{j}{2}} + h^3 \left( \sum_{j=0}^k \beta_{\frac{j}{2}}(x)f_{n+\frac{j}{2}} \right), \tag{1.2}$$

The method has order of accuracy of order five. The focus of the present work is to provide method with improved order of accuracy.

## 2 Derivation of the Block Method

This paper focused on direct solution of initial value problems of general third order ordinary differential equations of the type:

$$y''' = f(x, y, y', y''), y(0) = y_0, y'(0) = y'_0, y''(0) = y''_0, f \in \mathfrak{R}^n[a, b], y \in \mathfrak{R}, x \in [a, b] \tag{2.1}$$

using power series approximation solution of the form

$$y(x) = \sum_{j=0}^{(c+i)-1} a_j x^j, \tag{2.2}$$

where  $a_j, j = 0, 1, 2, \dots, k$  are the coefficients to be determined.

The third derivative of equation (5) is obtained as:

$$y'''(x) = \sum_{j=3}^k j(j-1)(j-2)a_j x^{j-3} \tag{2.3}$$

Combining (4) and (6) yield the differential system:

$$f(x, y(x), y'(x), y''(x)) = \sum_{j=3}^k j(j-1)(j-2)a_j x^{j-3} \tag{2.4}$$

Interpolate equation (5) at  $x = x_{n+j}, j = \frac{3}{2}, 2, \frac{5}{2}$  and collocate equation (6) at  $x = x_{n+j}, j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$  to have a set of linear equation of

$$AX = B \tag{2.5}$$

where

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdot & \cdot & \cdot & x_{1n} \\ x_{21} & x_{22} & \cdot & \cdot & \cdot & \cdot \\ x_{31} & x_{32} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{n1} & x_{n2} & \cdot & \cdot & \cdot & x_{nn} \end{bmatrix}, A = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix}, B = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix} \quad (2.6)$$

Solving (7) for A and substituting the value of  $a_j$ s in (5) yields the following continuous schemes after some simplifications:

$$y(t) = \alpha_{\frac{3}{2}} y_{n+\frac{3}{2}} + \alpha_2 y_{n+2} + \alpha_{\frac{5}{2}} y_{n+\frac{5}{2}} + h^3 \left( \sum_{i=0}^k \beta_{\frac{i}{2}} f_{n+\frac{i}{2}} \right) \quad (2.7)$$

Where

$$\alpha_{\frac{3}{2}} = 2t^2 - 9t + 10$$

$$\alpha_2 = -4t^2 + 16 - 15$$

$$\alpha_{\frac{5}{2}} = 2t^2 - 7t + 6$$

$$\beta_0 = \frac{1}{7257600} h^3 (1280 t^9 - 20160 t^8 + 134400 t^7 - 493920 t^6 + 1091328 t^5 - 1481760 t^4 + 1209600 t^3 - 551055 t^2 + 115147 t - 3930)$$

$$\beta_{\frac{1}{2}} = -\frac{1}{1209600} h^3 (1280 t^9 - 19200 t^8 + 119040 t^7 - 389760 t^6 + 701568 t^5 - 604800 t^4 + 442860 t^3 - 327713 t^2 + 77970)$$

$$\beta_1 = \frac{1}{483840} h^3 (1280 t^9 - 18240 t^8 + 105216 t^7 - 309792 t^6 + 471744 t^5 - 302400 t^4 - 53049 t^3 + 221359 t^2 - 114810)$$

$$\beta_{\frac{3}{2}} = -\frac{1}{362880} h^3 (1280 t^9 - 17280 t^8 + 92928 t^7 - 249984 t^6 + 341376 t^5 - 201600 t^4 + 135726 t^3 - 290177 t^2 + 211530)$$

$$\beta_2 = \frac{1}{483840} h^3 (1280 t^9 - 16320 t^8 + 82176 t^7 - 206304 t^6 + 266112 t^5 - 151200 t^4 - 26787 t^3 + 196771 t^2 - 175290)$$

$$\beta_{\frac{5}{2}} = -\frac{1}{1209600} h^3 (1280 t^9 - 15360 t^8 + 72960 t^7 - 174720 t^6 + 217728 t^5 - 120960 t^4 + 26160 t^3 - 9473 t^2 + 2370)$$

$$\beta_3 = \frac{1}{7257600} h^3 (1280 t^9 - 14400 t^8 + 65280 t^7 - 151200 t^6 + 184128 t^5 - 100800 t^4 + 19635 t^2 - 953 t - 3930)$$

Evaluating (9) at  $t = 0, \frac{1}{2}, 1, 3$  yield the discrete three step formulas.

$$y_n - 10 y_{n+\frac{3}{2}} + 15 y_{n+2} - 6 y_{n+\frac{5}{2}} = \frac{h^3}{241920} \left( -131 f_n - 15594 f_{n+\frac{1}{2}} - 57405 f_{n+1} - 141020 f_{n+\frac{3}{2}} - 87645 f_{n+2} - 474 f_{n+\frac{5}{2}} - 131 f_{n+3} \right) \quad (2.8)$$

$$y_{n+1/2} - 3 y_{n+\frac{5}{2}} + 8 y_{n+2} - 6 y_{n+\frac{3}{2}} = \frac{h^3}{241920} \left( 61 f_n - 744 f_{n+\frac{1}{2}} - 12819 f_{n+1} - 63464 f_{n+\frac{3}{2}} - 43689 f_{n+2} - 240 f_{n+\frac{5}{2}} - 65 f_{n+3} \right) \quad (2.9)$$

$$y_{n+1} - y_{n+\frac{5}{2}} + 3 y_{n+2} - 3 y_{n+\frac{3}{2}} = \frac{h^3}{241920} \left( 31 f_n - 249 f_{n+\frac{1}{2}} + 654 f_{n+1} - 15866 f_{n+\frac{3}{2}} - 14781 f_{n+2} + 3 f_{n+\frac{5}{2}} - 32 f_{n+3} \right) \quad (2.10)$$

$$y_{n+3} - 3 y_{n+\frac{5}{2}} + 3 y_{n+2} - y_{n+\frac{3}{2}} = \frac{h^3}{241920} \left( 32 f_n - 255 f_{n+\frac{1}{2}} + 921 f_{n+1} - 1774 f_{n+\frac{3}{2}} + 16986 f_{n+2} + 14109 f_{n+\frac{5}{2}} + 221 f_{n+3} \right) \quad (2.11)$$

In line with Kayode *et al.* (2014), the normalized form of the general block method is given by

$$Ay_i = Ey_n + h^{N-p} df(y_n) + h^{N-p} BF(y_n) \quad (2.12)$$

By combining the equations, the first and the second derivatives of the schemes are solved using Maple 17 application and the value of  $y_{n+\frac{1}{2}}, \dots, y_{n+3}, y'_{n+\frac{1}{2}}, \dots, y'_{n+3}, y''_{n+\frac{1}{2}}, \dots, y''_{n+3}$  are obtained as follows:

$$h^a \sum_{s=0}^q \phi_{m,s} y_{n+s}^e = h^e \sum_{s=0}^q \nabla_{m,s} y_n^p + h^{p-e} \left( \sum_{s=0}^q \nabla_{m,s} f_n + \sum_{s=0}^q \delta_{m,s} f_{n+s} \right) \quad (2.13)$$

where a is the power of the derivation of the continuous method and p is the order of the problem to solve. Equation (15) is solved for  $s = 0, \frac{1}{2}, 1, \dots, 3$  to obtain the following proposed third-step hybrid multistep method:

$$y_{n+\frac{1}{2}} = y_n + \frac{1}{2} y'_n h + \frac{1}{8} y''_n h^2 + \frac{h^3}{29030400} \left( 343801 f_n + 506604 f_{n+\frac{1}{2}} - 494715 f_{n+1} + 414160 f_{n+\frac{3}{2}} - 226605 f_{n+2} + 71364 f_{n+\frac{5}{2}} - 9809 f_{n+3} \right)$$

$$y_{n+1} = y_n + y'_n h + \frac{1}{2} y''_n h^2 + \frac{h^3}{226800} \left( 13774 f_n + 35976 f_{n+\frac{1}{2}} - 24465 f_{n+1} + 20800 f_{n+\frac{3}{2}} - 11370 f_{n+2} + 3576 f_{n+\frac{5}{2}} - 491 f_{n+3} \right)$$

$$y_{n+\frac{3}{2}} = y_n + \frac{3}{2} y'_n h + \frac{9}{8} y''_n h^2 + \frac{h^3}{358400} \left( +52893 f_n + 172692 f_{n+\frac{1}{2}} - 72495 f_{n+1} \right. \\ \left. +80640 f_{n+\frac{3}{2}} - 44145 f_{n+2} + 13932 f_{n+\frac{5}{2}} - 1917 f_{n+3} \right)$$

$$y_{n+2} = y_n + 2 y'_n h + 2 y''_n h^2 + \frac{h^3}{14175} \left( 3863 f_n + 13992 f_{n+\frac{1}{2}} - 3390 f_{n+1} + 6800 f_{n+\frac{3}{2}} \right. \\ \left. -3255 f_{n+2} + 1032 f_{n+\frac{5}{2}} - 142 f_{n+3} \right)$$

$$y_{n+\frac{5}{2}} = y_n + \frac{5}{2} y'_n h + \frac{25}{8} y''_n h^2 + \frac{h^3}{1161216} \left( 505625 f_n + 1945500 f_{n+\frac{1}{2}} - 256875 f_{n+1} \right. \\ \left. +1070000 f_{n+\frac{3}{2}} - 358125 f_{n+2} + 136500 f_{n+\frac{5}{2}} - 18625 f_{n+3} \right)$$

$$y_{n+3} = y_n + 3 y'_n h + \frac{9}{2} y''_n h^2 + \frac{h^3}{2800} \left( 1782 f_n + 7128 f_{n+\frac{1}{2}} - 405 f_{n+1} + 4320 f_{n+\frac{3}{2}} \right. \\ \left. -810 f_{n+2} + 648 f_{n+\frac{5}{2}} - 63 f_{n+3} \right) \quad (2.14)$$

$$y'_{n+\frac{1}{2}} = y'_n + \frac{1}{2} y''_n h + \frac{h^2}{483840} \left( 28549 f_n + 57750 f_{n+\frac{1}{2}} - 51453 f_{n+1} + 42484 f_{n+\frac{3}{2}} \right. \\ \left. -23109 f_{n+2} + 7254 f_{n+\frac{5}{2}} - 995 f_{n+3} \right)$$

$$y'_{n+1} = y'_n + y''_n h + \frac{h^2}{7560} \left( 1027 f_n + 3492 f_{n+\frac{1}{2}} - 1680 f_{n+1} + 1576 f_{n+\frac{3}{2}} - 873 f_{n+2} \right. \\ \left. +276 f_{n+\frac{5}{2}} - 38 f_{n+3} \right)$$

$$y'_{n+\frac{3}{2}} = y'_n + h^2 + \frac{3}{2} y''_n h + \frac{h^2}{17920} \left( 3795 f_n + 14850 f_{n+\frac{1}{2}} - 2403 f_{n+1} + 6300 f_{n+\frac{3}{2}} \right. \\ \left. -3267 f_{n+2} + 1026 f_{n+\frac{5}{2}} - 141 f_{n+3} \right)$$

$$y'_{n+2} = y'_n + 2 y''_n h + \frac{h^2}{945} \left( 272 f_n + 1128 f_{n+\frac{1}{2}} - 18 f_{n+1} + 656 f_{n+\frac{3}{2}} - 210 f_{n+2} + \right. \\ \left. 72 f_{n+\frac{5}{2}} - 10 f_{n+3} \right)$$

$$\begin{aligned}
 y'_{n+\frac{5}{2}} &= y'_n + 5/2 y''_n h + \frac{h^2}{96768} \left( 35225 f_n + 150750 f_{n+\frac{1}{2}} + 9375 f_{n+1} + 102500 f_{n+\frac{3}{2}} \right. \\
 &\quad \left. - 5625 f_{n+2} + 11550 f_{n+\frac{5}{2}} - 1375 f_{n+3} \right) \\
 y'_{n+3} &= y'_n + 3 y''_n h + \frac{h^2}{280} \left( 123 f_n + 540 f_{n+\frac{1}{2}} + 54 f_{n+1} + 408 f_{n+\frac{3}{2}} + 27 f_{n+2} \right. \\
 &\quad \left. + 108 f_{n+\frac{5}{2}} \right) \quad (2.15) \\
 y''_{n+\frac{1}{2}} &= y''_n + \frac{h}{120960} \left( 19087 f_n + 65112 f_{n+\frac{1}{2}} - 46461 f_{n+1} + 37504 f_{n+\frac{3}{2}} - 20211 f_{n+2} \right. \\
 &\quad \left. + 6312 f_{n+\frac{5}{2}} - 863 f_{n+3} \right) \\
 y''_{n+1} &= y''_n + \frac{h}{7560} \left( 1139 f_n + 5640 f_{n+\frac{1}{2}} + 33 f_{n+1} + 1328 f_{n+\frac{3}{2}} - 807 f_{n+2} \right. \\
 &\quad \left. + 264 f_{n+\frac{5}{2}} - 37 f_{n+3} \right) \\
 y''_{n+\frac{3}{2}} &= y''_n + \frac{h}{4480} \left( 685 f_n + 3240 f_{n+\frac{1}{2}} + 1161 f_{n+1} + 2176 f_{n+\frac{3}{2}} - 729 f_{n+2} \right. \\
 &\quad \left. + 216 f_{n+\frac{5}{2}} - 29 f_{n+3} \right) \\
 y''_{n+2} &= y''_n + \frac{h}{945} \left( 143 f_n + 696 f_{n+\frac{1}{2}} + 192 f_{n+1} + 752 f_{n+\frac{3}{2}} \right. \\
 &\quad \left. + 87 f_{n+2} + 24 f_{n+\frac{5}{2}} - 4 f_{n+3} \right) \\
 y''_{n+\frac{5}{2}} &= y''_n + \frac{h}{24192} \left( 3715 f_n + 17400 f_{n+\frac{1}{2}} + 6375 f_{n+1} + 16000 f_{n+\frac{3}{2}} + 11625 f_{n+2} \right. \\
 &\quad \left. + 5640 f_{n+\frac{5}{2}} - 275 f_{n+3} \right) \\
 y''_{n+3} &= y''_n + \frac{h}{280} \left( 41 f_n + 216 f_{n+\frac{1}{2}} + 27 f_{n+1} + 272 f_{n+\frac{3}{2}} + 27 f_{n+2} + 216 f_{n+\frac{5}{2}} \right. \\
 &\quad \left. + 41 f_{n+3} \right) \quad (2.16)
 \end{aligned}$$

### 3 Analysis of the Method

This section presented the analysis of the basic properties of the proposed third-step hybrid multistep method.

### 3.1 Order of the method

If  $y(x)$  is continuously differentiable, the according to [15] the linear operator associated with formular is defined by:

$$\alpha_{\frac{s}{2}}[y(x) : h] = y(x_n + \frac{s}{2}h) - (\sum_{a=0}^9 \alpha_a(\frac{s}{2})y^{(a)}(x)h^a - h^3 \sum_{a=0}^9 \beta_a(\frac{s}{2})y^{(3)}(x_n + \frac{s}{2}h), s = 1, 2, 3) \quad (3.1)$$

Taking  $y(x)$  as the valid solution of (1), the Taylor series expansion about the poinf  $x$  after using (19) give the formular for the local truncation error written as :

$$[\alpha_{\frac{s}{2}}y(x) : h] = C_0y(x) + C_1hy'(x) + C_2h^2y''(x) + \dots + C_{p+2}h^{p+2}y^{(p+2)}(x) + C_{p+3}h^{p+3}y^{(p+3)}(x) \quad (3.2)$$

The term  $C_{p+3}$  is called the error constant and implies that the local truncation error is given by

$$[\alpha_{\frac{s}{2}}[y(x) : h]] = C_{p+3}h^{p+3}y^{(p+3)}(x) + Oh^{p+4} \quad (3.3)$$

Since  $C_0 = C_1 = C_2 = \dots = C_{p+2} = 0, C_{p+3} \neq 0$ , refer to [16] then formular in (21) have order  $p = 7$  with error constant given by  $C_{p+3} = \frac{4001}{1857945600}$

### 3.2 Zero stability of the block

According to [17] a block is said to be zero stable if the root of the characteristic polynomial satisfy  $|z| \leq 1$  and the root of  $|z| = 1$  has multiplicity not exceeding the oder of the differential equation. Moreover, as  $h^n \rightarrow 0$ ,  $p(z) = z^{y-n}(\lambda - 1)$ , where n is the order of the differential equation, for the block ,  $r = 24, n = 3, \rho(z) = \lambda^{21}(\lambda - 1)^3 = 0$

This implies  $\lambda = 0, 1, 1, 1$

Therefore, the block method is zero stable.

### 3.3 Consistency of the method

From Eq.(13), the first and the second characteristic polynomial of the method are given by:

$$\rho(r) = r^3 - 3r^{\frac{5}{2}} + 3r^2 - r^{\frac{3}{2}}$$

$$\sigma(r) = \frac{32}{241920} - \frac{255}{241920}r^{\frac{1}{2}} + \frac{921}{241920}r^1 - \frac{1774}{241920}r^{\frac{3}{2}} + \frac{16986}{241920}r^2 + \frac{14109}{241920}r^{\frac{5}{2}} + \frac{211}{241920}r^3$$

This implies that the method presented in this paper is consistent since it satisfy the following conditions:

- (i). The order of the method is  $p = 7 \geq 1$ , which is obvious
- (ii). For the method,  $\alpha_1 = 1, \alpha_2 = -3, \alpha_3 = 3, \alpha_4 = -1$  . Hence  $\sum_{j=0}^4 a_j = 0$

(iii). If  $p(r) = r^3 - 3r^{\frac{5}{2}} + 3r^2 - r^{\frac{3}{2}}$  and

$$\rho'(r) = 3r^2 - \frac{15}{2}r^{\frac{3}{2}} + 6r - \frac{3}{2}r^{\frac{1}{2}}$$

$$\rho'''(r) = 6 - \frac{45}{8}r^{-\frac{1}{2}} + 0 + \frac{3}{8}r^{-\frac{3}{2}}$$

$$\therefore \rho'''(1) = \frac{3}{4} = 3!\sigma(1).$$

The conditions are satisfied , therefore the method is consistent.

### 3.4 Convergence of the method

According to [18] the necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable. Thus, it has been successfully shown from the above conditions, it could be seen that the method is convergent.

### 3.5 Region of absolute stability of the method

We consider the stability polynomial written in general form

$$\pi(r, \bar{h}) = \rho(r) - \bar{h}\sigma(r) = 0 \tag{3.4}$$

$\bar{h} = h^2\lambda$  and  $\lambda = \frac{\delta(f)}{\sigma(y)}$  is assumed constant. The stability polynomial of the formular (14) becomes:

$$y_{n+3} - 3y_{n+\frac{5}{2}} + 3y_{n+2} - y_{n+\frac{3}{2}} - \frac{h^3}{241920} \left( 32f_n - 255f_{n+\frac{1}{2}} + 921f_{n+1} - 1774f_{n+\frac{3}{2}} + 16986f_{n+2} + 14109f_{n+\frac{5}{2}} + 221f_{n+3} \right) = 0 \tag{3.5}$$

where

$$\rho(r) = r^3 - 3r^{\frac{5}{2}} + 3r^2 - r^{\frac{3}{2}} \text{ and}$$

$$\sigma(r) = \frac{32}{241920} - \frac{255}{241920}r^{\frac{1}{2}} + \frac{921}{241920}r - \frac{1774}{241920}r^{\frac{3}{2}} + \frac{16986}{241920}r^2 + \frac{14109}{241920}r^{\frac{5}{2}} + \frac{221}{241920}r^3$$

$$\bar{h} = \frac{\rho(r)}{\sigma(r)} \tag{3.6}$$

By inserting the value of  $\rho(r)$  and  $\sigma(r)$  into equation (25), we obtain boundary locus equation for the method as:

$$\bar{h}(r) = \frac{r^3 - 3r^{\frac{5}{2}} + 3r^2 - r^{\frac{3}{2}}}{\frac{32}{241920} - \frac{255}{241920}r^{\frac{1}{2}} + \frac{921}{241920}r - \frac{1774}{241920}r^{\frac{3}{2}} + \frac{16986}{241920}r^2 + \frac{14109}{241920}r^{\frac{5}{2}} + \frac{221}{241920}r^3} \tag{3.7}$$

Evaluate the imaginary part to zero, then we have

$$\bar{h}(\theta) = \frac{241920(10626 - 18414\cos\frac{\theta}{2} + 11880\cos\theta - 5555\cos\frac{3\theta}{2} + 1782\cos2\theta - 351\cos\frac{5\theta}{2} + 32\cos3\theta)}{491698284 + 4243036\cos\frac{\theta}{2} - 10299024\cos\theta + 16428274\cos\frac{3\theta}{2} - 5701404\cos2\theta - 112710\cos\frac{5\theta}{2} + 14144\cos3\theta}$$

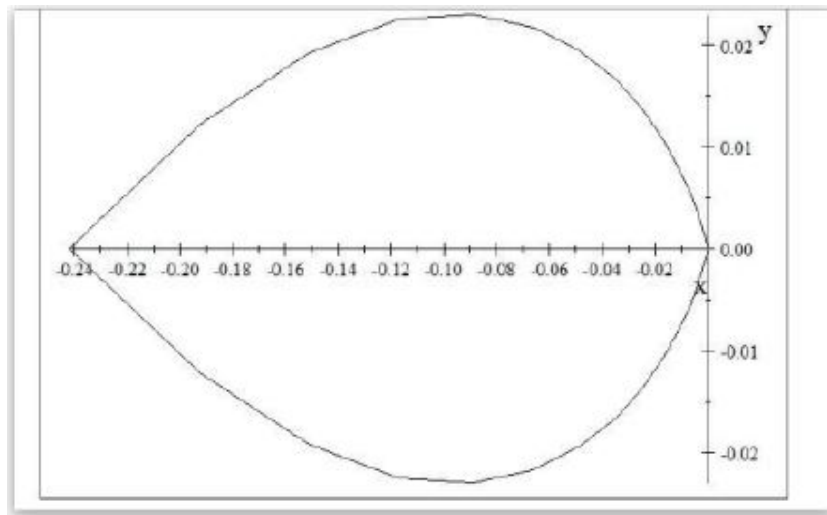


Fig. 1. Region of Absolute Stability of the proposed method



## 4 Numerical Application

The method was utilized to solve some specific initial value problems of third order ODE to verify its accuracy, workability, and applicability. The following notations are used to represent the current findings:

XVC: Value of the independent variable where a numerical value is taken

ERC: Exact result at XVC

NRC: Numerical result of XVC

ERR: Error in proposed method at XVC

The table showing  $Errc = |y_{ex} - y_c|$ , where  $y_{ex}$  is the exact solution,  $y_c$  is the computed result and  $A_c$  absolute error defined as  $Errc = |y_c - y_{ex}|$  was also reported.

### 4.1 Problem 1

$$y''' = -y,$$

$$y(0) = 1, y'(0) = -1, y''(0) = 1, h = \frac{1}{10}$$

$$\text{Exact solution: } y(x) = e^{-x}$$

Source: [19]

### 4.2 Problem 2

$$y''' + 4y' - x = 0, h = 0.1$$

$$y(0) = 0, y'(0) = 0, y''(0) = 1, h = \frac{1}{10}$$

$$\text{Exact solution: } y(x) = \frac{3}{16}(1 - \cos 2x) + \frac{1}{18}x^2$$

Source: [20]

### 4.3 Problem 3

$$y''' = e^x, h = 0.1$$

$$y(0) = 3, y'(0) = 1, y''(0) = 5, h = \frac{1}{10}$$

$$\text{Exact solution: } y(x) = 2 + 2x^2 + e^x$$

Source: [6]

**Table 1. Comparison of result obtained with proposed method and that of [19] on Problem 1,  $h = 0.1$**

XVR	ERC	NRC	ERR p=7,k =3	ERR in [19] p= 8
0.10	0.90483741803595957316	0.90483741803595837255	1.20061E-15	6.055156E-13
0.20	0.81873075307798185867	0.81873075307797625912	5.59955E-15	2.13841E-12
0.30	0.74081822068171786607	0.74081822068170467209	1.319398E-14	7.39575E-12
0.40	0.67032004603563930074	0.67032004603561636696	2.293378E-14	2.15813E-12
0.50	0.60653065971263342360	0.60653065971263335181	3.512021E-14	1.484579E-11
0.60	0.54881163609402643263	0.54881163609397670806	4.972457E-14	1.098521E-11
0.70	0.49658530379140951470	0.49658530379134358118	6.593352E-14	3.142886E-11
0.80	0.44932896411722159143	0.44932896411713766565	8.392578E-14	2.309530E-11
0.90	0.40656965974059911188	0.40656965974049548523	1.0362665E-13	5.154149E-11
1.00	0.36787944117144232160	0.36787944117131795034	1.2437126E-13	8.200535E-11

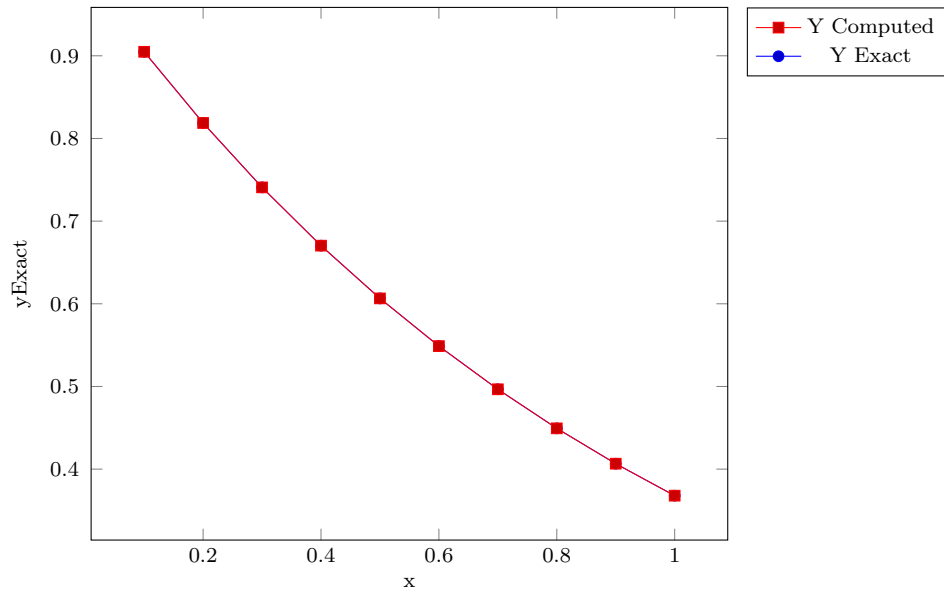


Fig. 2. Solution curve of problem 1 as compared with the exact solution

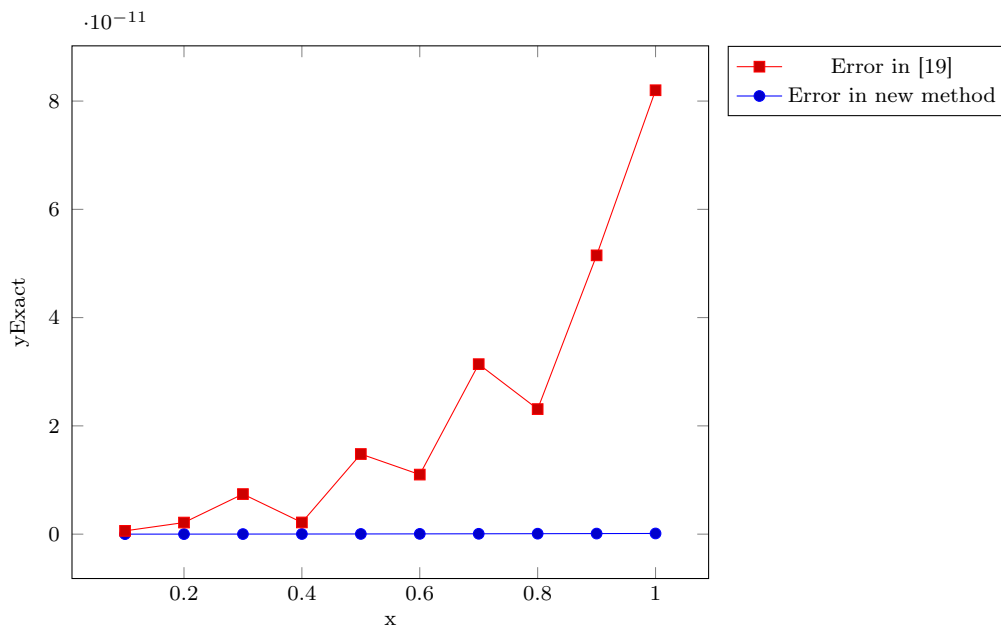
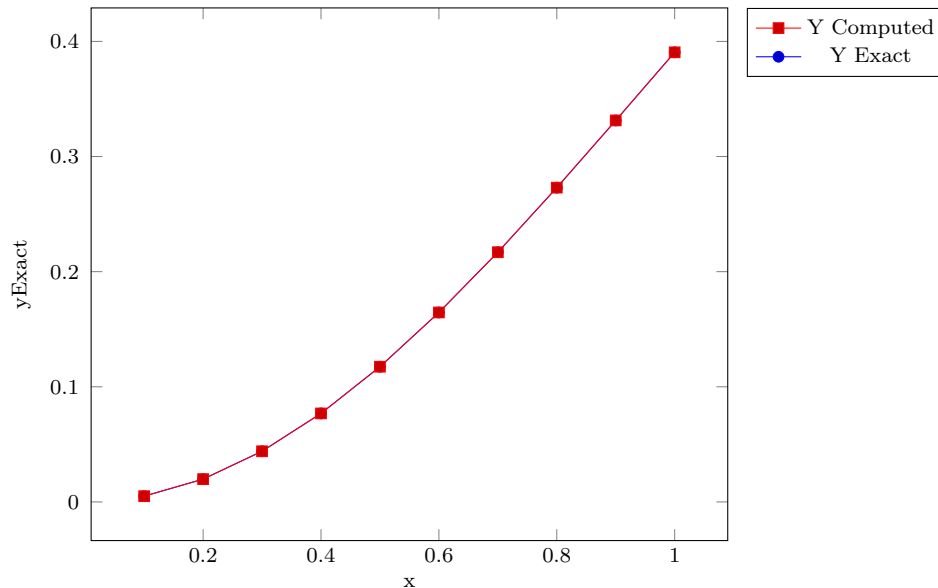


Fig. 3. Comparison of absolute errors of the proposed method on problem 1 as compared with [19]

**Table 2. Comparison of result obtained with proposed method and that of [20]  $h = 0.1$**

XVR	ERC	NRC	ERR in newly method, $k=3, p=7$	Error in [20], $p=7$
0.10	0.0049875166547671941600	0.0049875166545145588800	2.526352800E-13	1.6654E-08
0.20	0.019801063624459046980	0.019801063623291460450	1.167586530E-12	3.8095E-07
0.30	0.043999572204435319270	0.043999572201726883342	2.708435928E-12	1.5664E-06
0.40	0.076867491997406483580	0.076867491992833991841	4.572491739E-12	3.9865E-06
0.50	0.11744331764972380299	0.11744331764300377051	6.72003248E-12	7.9597E-06
0.60	0.16455792103562370419	0.16455792102655810685	9.06559734E-12	1.3680E-05
0.70	0.21688116070620482401	0.21688116069492027661	1.128454740E-11	2.1195E-05
0.80	0.27297491043149163616	0.27297491041831323226	1.317840390E-11	3.0396E-05
0.90	0.33135039275495382287	0.33135039274028196095	1.467186192E-11	4.10008E-05
1.00	0.39052753185258919756	0.39052753183702406999	1.556512757E-11	5.2605E-05



**Fig. 4. Solution curve of problem 2 as compared with the exact solution**

**Table 3. Comparison of result obtained with proposed method and that of [6],  $h = 0.1$**

x-value	y-exact-solution	y-computed solution	Error in newly method, $p=7$	Error in [6], $p=8$
0.10	3.1251709180756476248	3.1251709180756460575	1.5673E-15	2.531308E-14
0.02	3.3014027581601698339	3.3014027581601625265	7.3074E-15	1.612044E-13
0.03	3.5298588075760031040	3.5298588075759858842	1.72198E-14	4.023448E-13
0.40	3.8118246976412703178	3.8118246976412395954	3.07224E-14	7.536194E-13
0.50	4.1487212707001281468	4.1487212707000784350	4.97118E-14	1.212364E-12
0.60	4.5421188003905089749	4.5421188003904347878	7.41871E-14	1.780798E-12
0.70	4.9937527074704765216	4.9937527074703731589	1.033627E-13	2.456702E-12
0.80	5.5055409284924676046	5.5055409284923278058	1.397988E-13	2.212097E-11
0.90	6.0796031111569496638	6.0796031111567661698	1.834940E-13	5.231993E-11
1.00	6.7182818284590452354	6.7182818284588118470	2.333884E-13	8.86011E-11

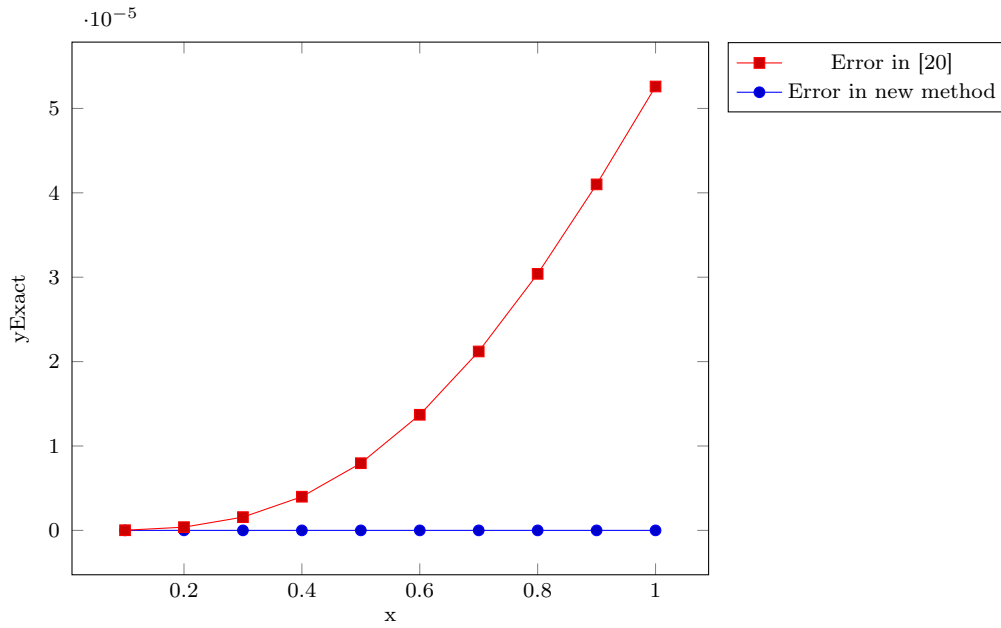


Fig. 5. Comparison of absolute errors of the proposed method on problem 2 as compared with [20]

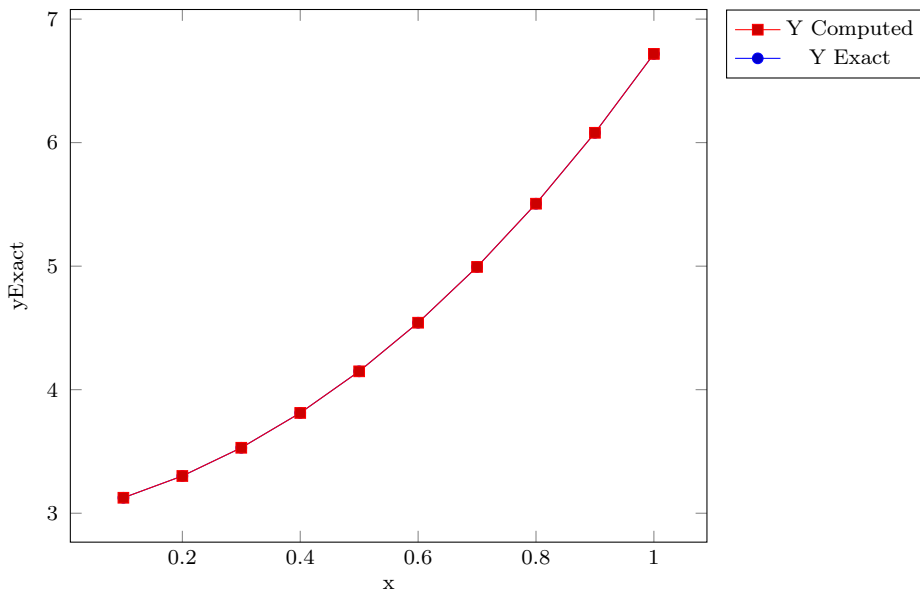


Fig. 6. Solution curve of problem 3 as compared with the exact solution

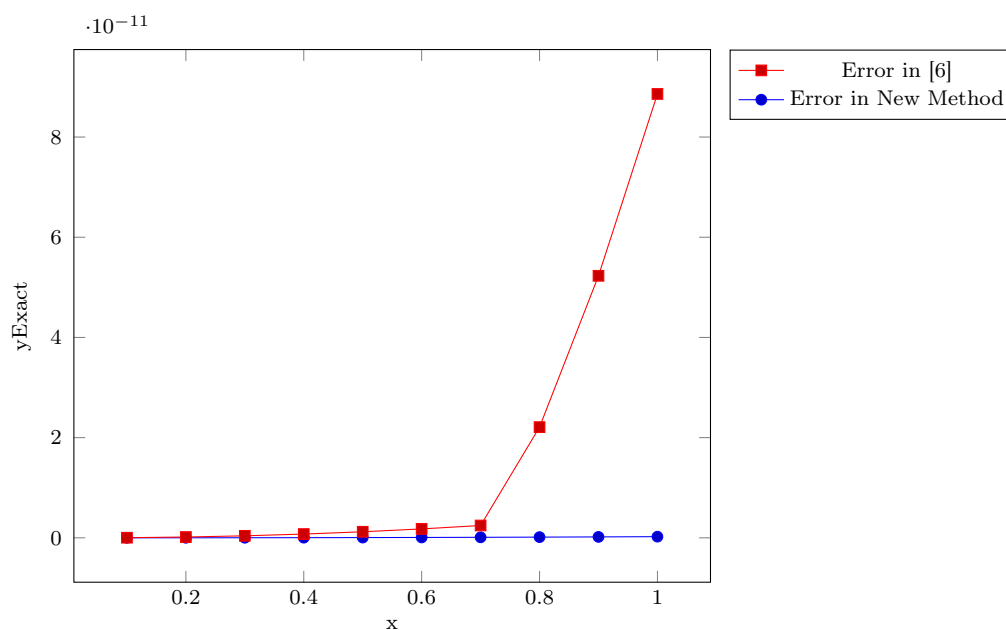


Fig. 7. Comparison of absolute errors of the proposed method on problem 3 as compared with [6]

#### 4.4 Problem 4

$y''' - y'(2xy'' + y') = 0,$   
 $y(0) = 0, y'(0) = 0, y''(0) = \frac{1}{2}, h = \frac{1}{10}$   
 Exact solution:  $y(x) = 1 + \frac{1}{2} \log\left(\frac{2+x}{2-x}\right)$   
 Source: [21]

Table 4. Comparative overview of methodological Error

x	yapprox	yExact	Error in new method k=3,p=7	Error in [21] p=9
0.1	1.050041729	1.050041729	1.59595E-16	1.93148E-08
0.2	1.100335348	1.100335348	2.91434E-16	5.60825E-07
0.3	1.151140436	1.151140436	7.49401E-16	3.75510E-06
0.4	1.202732554	1.202732554	1.44329E-15	1.34028E-05
0.5	1.255412812	1.255412812	2.9976E-15	3.25906E-05
0.6	1.309519604	1.309519604	6.32827E-15	5.81649E-05
0.7	1.365443754	1.365443754	1.28786E-14	7.15239E-05
0.8	1.42364893	1.42364893	2.63123E-14	2.56483E-05
0.9	1.484700279	1.484700279	5.66214E-14	1.70915E-04
1	1.549306144	1.549306144	1.27787E-13	6.70643E-04

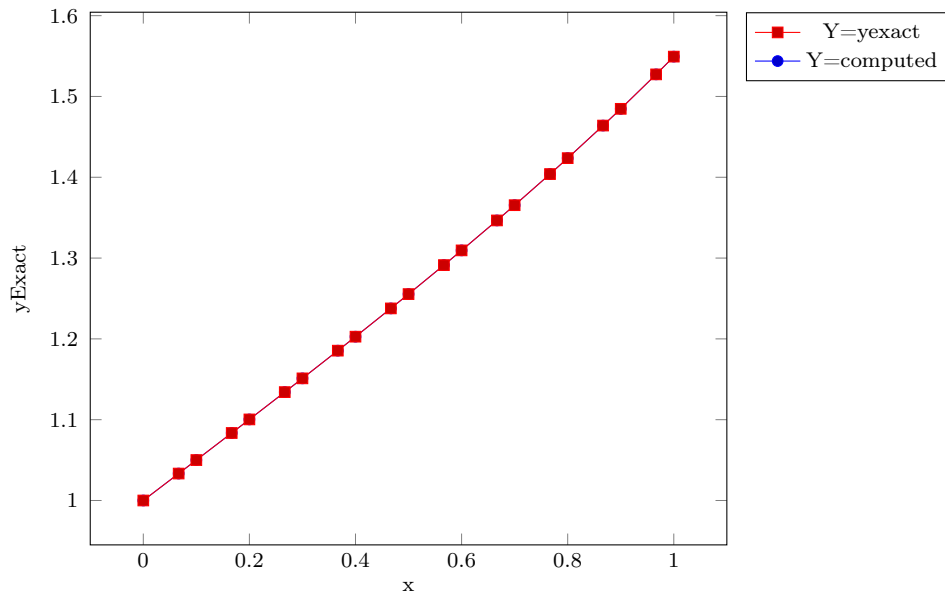


Fig. 8. Solution curve of problem 4 as compared with the exact solution

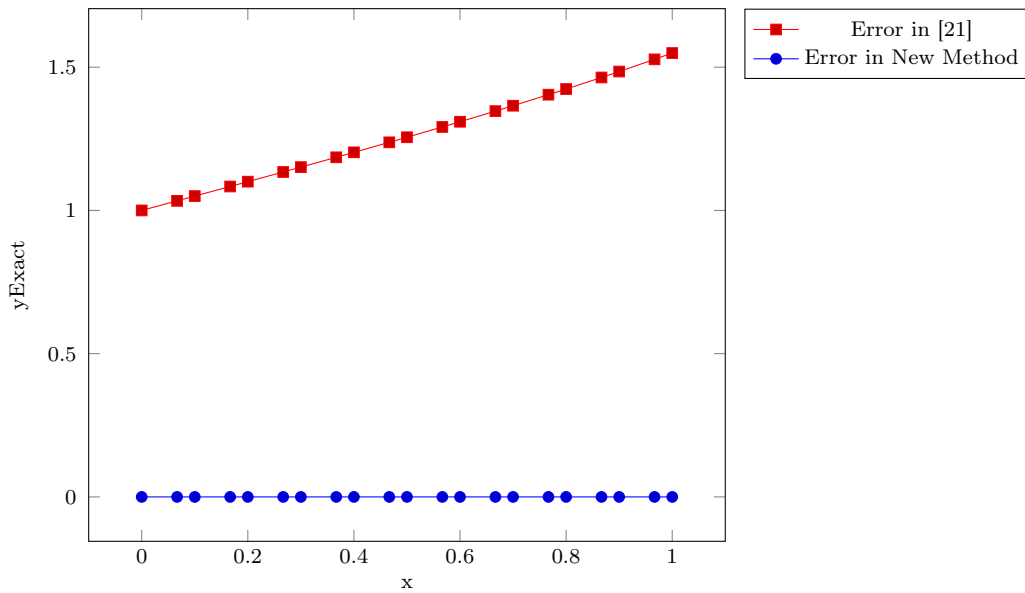


Fig. 9. Comparison of absolute errors of the proposed method on problem 4 as compared with [21]

## 5 Discussion

In this research work, we have applied the procedure of collocation and interpolation to develop linear hybrid multistep method for solving initial value problem of third order ordinary differential equations. In the Table 1, Table 2 and Table 3. It is shown that our new method is more accurate than the method proposed by [6], [19]-[20]. It has been seen from Fig. 2 to Fig. 9 that our new method yield better result than the results presented by [19]-[21],[6] despite the high order of their method. Our new method of order seven are more efficiency and accurate than his method of order nine.

## 6 Conclusion

The 3-step methods has been developed for the direct solution of third order ordinary differential equations. The methods were obtained from the same continuous methods derived via interpolation and collocation procedure. The stability properties and region of the methods were discussed. The methods were applied in block form. Numerical results obtained using the obtained block methods showed that they are efficient and adequate for general third order problems of ordinary differential equations. In fact, when the new technique result were compared to the block method proposed by [6]-[7],[19]-[20] the new method was more accurate.

## Competing Interests

Authors have declared that no competing interests exist.

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