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The Multi-fractal Spectrum Model for the Measurement of Random Behaviour of Asset Price Returns

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Abstract

To forecast the market risk, assessing the stock price indices is the foundation. Multi-fractal has lots of advantage when explaining the volatility of the stock prices. The asset price returns is a multi-period (multi-fractal dimension) market depending on market scenarios which are the measure points. This paper considers the multi-fractal spectrum model (MSM) to measure the random character of asset price returns, aimed at deriving the MSM version of the random behaviour of equity returns of the existing ones in literature. We investigate the rate of returns prior to market signals corresponding to the value for packing dimension in fractal dispersion of Hausdorff measure. Furthermore, we give some conditions which determine the equilibrium price, the future market price and the optimal trading strategy.

Keywords: Multi-fractal spectrum model, assets price returns, hausdorff measure optimal trading strategy.

1 Introduction

The multifractal formalism was introduced in the context of fully-developed turbulence data analysis and modeling to account for the experimental observation of some deviation to Kolmogorov theory (K41) of homogenous and isotropic turbulence [1]. The predictions of various multiplicative cascade models, including the weighted curdling (binomial) model proposed by [2] were tested using box-counting (BC) estimates of the so-called $f(\alpha)$ singularity spectrum of the dissipation field [3]. Alternatively, the intermittent nature of the velocity fluctuations were investigated via the computation of the $D(h)$ singularity spectrum using the structure function (SF) method. In financial economics, multi-fractal spectrum model (MSM) has been used to analyze the pricing implications of multi-frequency risk .The models have had some success in explaining the excess volatility of stock returns compared to fundamentals and the negative skewness of equity returns. They have also been used to generate multi-fractal jump–diffusion [4].

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The multifractal formalism of multi-affine functions amounts to compute the so-called singularity spectrum D(h) defined as the Hausdorff dimension of the set where the Hölder exponent is equal to h. Once one has the function, h(t),characterizing the Holder exponent , one can construct the singularity spectrum . It is believed that the singularity spectrum is a synonym to the multi- fractal spectrum.

The specifics of how to calculateh (t) and D(α) in practice vary from approximating them outright by their definition or by using wavelets to approximate the Holder exponent and then using a Legendre transform to approximate the multi-fractal spectrum.

MSM is a stochastic volatility model [5,6] with arbitrarily many frequencies. MSM builds on the convenience of regime –switching models, which were advanced in economics and finance [7]. MSM is closely related to the multi-fractal model of Asset Returns (MMAR) [8]. MSM improves on the MMAR's combinatorial construction by randomizing arrival times, guaranteeing a strictly stationary process. MSM provides a pure regime-switching formulation of multi-fractal measures, which were pioneered by [9,10,11].

Extensions of MSM to multiple assets provide reliable estimates of the value-at-risk in a portfolio of securities [12]. MSM often provides better volatility forecasts than some of the best traditional models both in and out of sample [13]. Report considerable gains in exchange rate volatility forecasts at horizons of 10 to 50 days as compared with GARCH (1,1), Markov-switching GARCH, [14,15] and Fractionally Integrated GARCH [16]. [17] obtained similar results using linear predictions. Building on the multifractal formalism presented earlier, MMAR represents a simple compound process that has a closed-form multifractal spectrum. MMAR produces volatility clustering, heavy tails and long memory in returns.

Our aim in this paper is to apply some tools of multifractal analysis to the measurement of random behaviour of the stock market price returns. The problem associated with random behaviour of stock exchange has been addressed extensively by many authors [18,19]. [20] among others followed the traditional approach to pricing options on stocks with stochastic volatility which starts by specifying the joint process for the stock price and its volatility risk. Their models are typically calibrated to the prices of a few options or estimated from the time series of stock prices. On the other hand, [21] considered astochastic model of price changes at the floor of stock market. Here the equilibrium price and the market growth rate of shares were determined. There have been some works with considerable extensions and constrains subsequently [22,23]. In this paper we present a dynamic stochastic multi-fractal spectrum model of variation of the stock returns aimed at deriving the MSM version of the random behaviour of equity returns of the existing ones in literature. The equilibrium price and growth rate of an asset, using a linear rate of return, r_t are determined. We first developed a method of computing the so called singularity spectrum which is equivalent to computing the Holder exponent. Further more, the optimal trading strategy is determined.

2 Basic Tools and Preliminaries

Let $(P_t)_{t\geq 0}$ denote the price process of a security, in particular of a stock. To allow comparison of investments in different securities, it is necessary to investigate the rates of return defined by

$$Z_t = \ln P_t - \ln P_{t-1} \tag{1}$$

Most authors prefer these rates, which correspond to continuous compounding, to the alternative

$$Y_t = \frac{(P_t - P_{t-1})}{P_{t+1}} \tag{2}$$

The reason for this is that the return over n periods, for example n days, is then just the sum;

$$Z_t + \cdots + Z_{t-n-1} = \ln P_{t+n-1} - \ln P_{t-1}. \tag{3}$$

As a model for stock prices the natural candidate is now the multi-fractal process $\left(X_t^{d,n}\right)_{t\geq 0}$

$$dY_t = pY_t dt + Y_t dZ_t^{d,n} \tag{4}$$

This can also be written in the form $dY_t = Y_t dZ_t$ with $Z_t = X_t^{d,n}$. The solution of this equation is the Doleans-Dade exponential [24].

$$Y_t = Y_0 exp\left(X_t^{d,n}\right) \prod_{0\leq j\leq t}\left(1 + \Delta X_t^{d,n}\right) e^{-\Delta X_t^{d,n}} \tag{5}$$

The MSM model can be specified in both discrete time and continuous time.

2.1 Discrete Time

Let $P_t$denote the price of a financial assets, and let

$$Z_t = ln(P_t/P_{t-1}) \tag{6}$$

denote the returns over two consecutive periods. In MSM, returns are specified as

$$r_t = \mu + \bar{\sigma}\left(M_{1,t}M_{2,t} \dots M_{\bar{k},t}\right)^{1/2}\xi_t, \tag{7}$$

Where μ and σ are constants and $\{\xi_t\}$ are independent standard Gaussians. Volatility is driven by the first-order latent Markov state vector:

$$M_t = \left(M_{1,t}M_{2,t} \dots M_{\bar{k},t}\right)\epsilon R_+^{\bar{k}}. \tag{8}$$

Given the volatility state $M_t$the next-period multiplier $M_{k,t+1}$ is drawn from a fixed distribution M with probability $\gamma_k$and is otherwise left unchanged $M_{k,t}$ drawn from distribution M with probability γ_k. with probability $1 - \gamma_k$

$$M_{k,t} = M_{k,t-1}. \tag{9}$$

The transition probability is specified by

$$\gamma_k = 1 - (1 - \gamma_1)^{(b^{k-1})} \tag{10}$$

The sequence γ_k is approximately geometric with $\gamma_k \approx \gamma_1^{b^{k-1}}$ at low frequency. The marginal distribution M has a unit mean, has a positive support, and is independent of K. In empirical applications, the distribution M is often a discrete distribution that can take the values m_0 or $2 - m_0$ with equal probability. The return process r_t is then specified by the parameters $\theta = (m_0, \mu, \bar{\sigma}, b, \gamma_1)$, Note that the number of parameters is the same for all $\bar{k} > 1$.

2.2 Continuous Time

MSM is similarly defined in continuous time. The price process follows the diffusion:

$$\frac{dp_t}{p_t} = \mu dt + \sigma(M_t) dW_t, \tag{11}$$

where $\sigma(M_t) = \bar{\sigma}\left(M_{1,t} \dots M_{\bar{k},t}\right)^{1/2}, W_t$ is a standard Brownian motion, and μ and $\bar{\sigma}$ are constants. Each component follows the dynamics: $M_{k,t}$ drawn from distribution M with probability $\gamma_k dt$. $M_{k,t+dt} = M_{k,t}$ with probability $1 - \gamma_k dt$. The intensities vary geometrically with $K{:}\,\gamma_k = \gamma_1^{b^{k-1}}$. When the number of components $\bar{k}$ goes to infinity, continuous-time MSM converges to a multi-fractal diffusion , whose sample paths take a continuum of local Holder exponents on any finite time interval.

When M has a discrete distribution, the Markov state vector M_t takes finitely many values $m^1, \dots, m^d \in R_+^{\bar{k}}$. For instance, there are $d = 2^{\bar{k}}$ possible states in binomial MSM. The Markov dynamics are characterized by the transition matrix $A = \left(a_{i,j}\right)_{1 \le i,j \le d}$ with components;

$$a_{i,j} = P(M_{t+1} = m^j | M_{tt} = m^i). \tag{12}$$

Conditional on the volatility state, the return r_t has Gaussian density

$$f(r_t | M_t = m^i) = \frac{1}{\sqrt{2\pi\sigma^2(m^i)}} exp\left[-\frac{(r_t - \mu)^2}{2\sigma^2(m^i)}\right]. \tag{13}$$

Furthermore, $M_t = M_{1t}, M_{2t}, \dots, M_{kd}$ is the multiplicative measure defined at different stages. In fact, $M_B(0 \le b \le b - 1)$ are the non-negative multipliers with arbitrary distribution. The product of k multiplier is defined by [25];

$$\mathcal{M}(\Delta t) = M_{1t} M_{2t} \dots M_{kt}, \tag{14}$$

with the scaling relationship $\mathbb{E}\left[\left(\mathcal{M}(\Delta t)\right)^q\right] = [\mathbb{E}(M)^q]$ or

$$[\mathbb{E}(M)^q] = (\Delta t)^{\tau(q)+1}, \tag{15}$$

where $\tau(q) = -\log_b \mathbb{E}(M)^q - 1$.

If the latest observation P_t of the designated process at time $t > d$ is conditioned on the information;

$$\varphi_{t-1} = \{P_u - \mu_u : u = 1,2, \dots, t-1\} \tag{16}$$

available up to time $t-1$, then

$$P_t|\varphi_{t-1} \sim (\mu_t, \sigma^2(M_t)) \tag{17}$$

where (using (10)),

$$\sigma^2(M_t) = (\Delta t)^{\tau(q)+1} h(\xi_{t-1}, \xi_{t-2}, \dots, \xi_{t-d}, \alpha) \tag{18}$$

and

$$\xi_u = P_u - \mu_u, u = 1,2, \dots, t-1. \tag{19}$$

The function $h(.)$ with parameters $\alpha = \{\alpha_k : k = 0,1, \dots, d\}$ is defined as

$$h(\xi_{t-1}, \xi_{t-2}, \dots, \xi_{t-d}, \alpha) = \alpha_0 + \sum_{k=1}^{d} \alpha_k \xi_{t-k}^2, \tag{20}$$

so that given the history (18) of the process up to time $t-1$, the conditional distribution of P_t is normal with mean μ_t and variance

$$\sigma^2(M_t) = (\Delta t)^{\tau(q)+1} \left(\alpha_0 + \sum_{k=1}^{d} \alpha_k (P_{t-k} - \mu_{t-k})^2\right). \tag{21}$$

Formally, if the price of an asset at time t is P_t for $t \in [0, T]$ and $T < \infty$, then the associated log-price process is given by

$$Z_t = \ln P_t - \ln P_0, \quad t \in [0, T].$$

The log-price process is then modeled as a compound process

$$Z_t = W[\theta(t)], t \in [0, T],$$

where $W(t)$ is Brownian motion and $\theta(t)$ is a stochastic process termed trading time (which is the cumulative distribution function (c.d.f.) of a random multifractal measure, and the processes $W(t)$ and $\theta(t)$ are assumed to be independent.

The fractal dimension is the basic notion for describing structures that have a scaling symmetry. Scaling symmetry means self-similarity of the considered object on varying scale of magnification.

[26] introduced the first notion of dimension, providing a measure for filling space which allows for the possibility of non-integral dimensions. Then D-dimensional Hausdorff measure on a set A is given by

$$M_D(A) = \lim_{\substack{e \to 0 \\ r_k < \in}} \sum_k r_k^D, D > 0. \tag{22}$$

Define the optimal covering of this set using spheres of variable radius r_k. The Hausdorff-dimension D_H is the value of D at which M_D jumps from ∞ to 0, while the dimension D can be calculated as

$$D = \frac{\log N}{\log S}, \tag{23a}$$

(N is the number of small pieces that go into the larger one and S is the scale to which the smaller pieces compare to the larger one). Equivalently for a given precision level $\varepsilon > 0$, $N(\varepsilon)$ satisfies a power law as $\varepsilon \to 0$ so that

$$N(\varepsilon) \sim \varepsilon^{-D}. \tag{23b}$$

In equation (23b), D is a constant called the fractal dimension, which helps to analyze the structure of a fixed multifractal. For a large class multifractals, the dimension $D(\alpha)$ coincides with the multifractal spectrum. For any $\alpha \geq 0$, the set $T(\alpha)$ can be defined as the Hölder exponent α with a fractal dimension$D(\alpha)$ satisfying $0 \leq D(\alpha) \leq 1$.

Consider the set A given by $A = \{a: 0 \leq D \leq b-1\}$. That is we can as well consider non-negative multipliers $M_D(0 \leq D \leq b-1)$ with arbitrary distribution [27]. If the multipliers are identically distributed ($M_D \underline{d} M \forall D$), and independent at different stagies of the construction, the limit multiplicative measure is so called conservative when mass is conserved exactly at each stage. Thus $\mathbb{E}(\Sigma M_D) = 1$ or equivalently $M = {}^{1}\!/_{b}$.

If A is any set in$\mathbb{R}^n$, the Hausdorff measure of a subset A of $\mathbb{R}^n$ is defined by

$$h - m(A) = \lim_{\delta \to 0}\left\{\sum_{A \subseteq c_i} \inf h\big(d(c_i)\big)\right\}, \tag{24}$$

where the infimum extends over all countable covers of A by sets c_i of diameter $d(c_i) < \delta, h$ is monotone, right continuous and $h(0^+) = 0$.

Let μ be a finite Borel measure in $\mathbb{R}^n$, consider the spherical density of U at$x$ given by

$$D_h(u, x) = \lim\ sup_{r \to 0} \frac{u\big(B(x,r)\big)}{h(r)}, \tag{25}$$

where $B(x, r)$ is a ball of radius r and center x. The behavior of this density on a set $E \subset \mathbb{R}^n$ as a function of x connects $h(E)$ with the size of E in some sense. For example, in (25), we have that if

$$\lim_{r \to 0} \frac{u(E \cap) B(x,r)}{h(r)} \leq K,\ \forall B(x, r) \in E, \tag{26}$$

then

$$h - m(E) \geq \lambda k^{-1} \mu(E), \tag{27}$$

where λ is a constant [28,29].

In order to apply the density on random sets, we need to construct a suitable Borel finite measure $\cup$ on $\mathbb{R}^n$ with support in$E$ using the sample path of a process .It is natural to try a measure which is uniformly spread over E, so an obvious candidate for$E(w) = X_t^{d,n}$ where $X_t^{d,n}$ is a stochastic process in $\mathbb{R}^n$ of index d, is the occupation measure defined by

$$\begin{aligned} u\big(B(0,r)\big) &= \left|\left\{E\epsilon\mathbb{R}^n : X_t^{d,n}\epsilon B(0,r)\right\}\right| \\ &= \int_0^\infty I_{B(0,r)}\, X_t^{d,n} dt \\ &= T(r). \end{aligned} \tag{28}$$

Equation (28) is called the sojourn time and I is the indicator function.

This density theorem is necessary because the definition of Hausdorff measure (25) requires us to consider all possible coverings of E by small sets in order to show that $h - m(E) > 0$.

Hence to examine random sets using Hausdorff measure, then important tool is $T(r)$ process, i.e. time spent in a ball of radius r. Thus (25) which becomes

$$Limsup\,\frac{T(r)}{h(r)} \tag{29}$$

is relevant to Hausdorff measure.

Let $(\mathbb{R}^n, \beta(\mathbb{R}^n))$ be a measurable space and let $f: \beta(\mathbb{R}^n) \to \mathbb{R}$ be a measurable function. Let λ be a real valued function defined on $\beta(\mathbb{R}^n)$. Then the multi- fractal spectrum with respect to the function $f$and $\lambda$is defined by

$$D(\alpha) = \lambda\{x\epsilon\beta(\mathbb{R}^n) : f(x) = \alpha\}. \tag{30}$$

In multi-fractal analysis, the function $D(.)$ is usually taken to be the Hausdorff dimension [30,31]. The basic problem is to calculate the function $D(\alpha)$. To do this, we need to know what the function $f(x)$ is. [31] Defined

$$f(x) = \lim\ sup\,\frac{\cup\big(B(x,r)\big)}{r^\alpha(\log r)} \tag{31}$$

and showed that

$$\dim E = \left\{x \in \mathbb{R}^n : Limsup_{t\to 0}\frac{\cup\big(B(x,r)\big)}{r^2|\log r|} = \alpha\right\} = 2 - \frac{aq_n^2}{2}$$

for the Brownian motion X_t in $\mathbb{R}^n, n \geq 3$ for all $0 \leq \alpha \leq \frac{4}{q_n^2}$ and q_n is the first position zero of the Bessel function $J_n/_{2-2(x)}$.

On the other hand, [32] defined another gauge function given by

$$f(x) = \lim\ sup\,\frac{\cup\big(B(x,r)\big)}{r^2(\log r)^\lambda}, \tag{32}$$

and show that if $\lambda > 1$ then

$$\dim E = \left\{x\epsilon\mathbb{R}^n : \lim sup \frac{\cup(B(x,r))}{r^2|\log r|^{\lambda}} = 0\right\} = 2\,, a.s.\ n > 3. \tag{33}$$

The multifractal formalism of multi-affine functions amounts to compute the so-called *singularity spectrum*$D(\alpha)$ defined as the Hausdorff dimension of the set where the Hölder exponent is equal to α [33].

To obtain the function $f(x) = \lim\, \sup \frac{T(r)}{h(r)}$ in our case, we require the local asymptotic behavior of the sample path of the process.And what comes to mind is the subject of the law of iterated logarithm (LIL).

To this end we assume a double stochastic integrals by a direct adaptation of the case of the Brownian motion and set

$$h(t) = 2t \log log \frac{1}{t} \text{ for } t > 0. \tag{34}$$

In what follows, we now state;

Theorem 1:

For $t > 0$ and $h(t) = 2t \log log \frac{1}{t}$, the so-called *singularity spectrum* $D(\alpha)$ defined as the Hausdorff dimension of the set where the Hölder exponent is equal to α is given by

$$\lim_{t\to 0} sup \frac{X_t^d}{h(t)} = \frac{(1+\gamma)^2}{2\theta}, \theta = 1, \gamma\epsilon[0,1]. \tag{35}$$

Proof

Let d be a predictable process valued in a bounded interval $[\alpha_0, \alpha_1]$ for some real parameters $0 \le \alpha_0 \le \alpha_1$, and $X_t^d := \int_0^t \int_0^u a_r dW_r dW_u$.

Then $\alpha_0 \le lim \frac{2X_t^d}{h(t)} \le \alpha_1 a.s.$

Now set $\bar{\alpha} = \frac{(\alpha_0+\alpha_1)}{2} \ge 0$

and

$$\delta := \frac{(\alpha_1+\alpha_0)}{2}.$$

For the first inequality, we have by the law of the iterated logarithm for the Brownian motion.

$$\bar{\alpha} = \lim_{t\to 0} sup \frac{2X_t^d}{h(t)} \le \delta \lim_{t\to 0} sup \frac{2X_t^d}{h(t)} + \lim_{t\to 0} sup \frac{2X_t^d}{h(t)},$$

where $\check{d} = \delta^{-1}(\bar{\alpha} - d)$ is the value in$[-1,1]$. It then follows from the second inequality that;

$$\lim_{t\to 0} sup \frac{2X_t^d}{h(t)} \geq \bar{\alpha} - \delta = \beta_0.$$

For the prove of the second inequality, we assume without loss of generality that $\|d\|_\infty \leq 1$.

Let $T > 0$ and $\lambda > 0$ be such that $2\lambda T < 1$. Then from Doob's maximal inequality for sub-martingales for all $\beta \geq 0$, we have

$$P\left[\max_{0\leq t\leq T} 2X_t^d \geq \beta\right] = P\left[\max_{o\leq t\leq T} exp\,(2\lambda X_t^d) \geq exp(\lambda\beta)\right]$$

$$\leq e^{-\lambda\beta} E\left[e^{2\lambda X_T^d}\right]$$
$$= e^{-\lambda(\beta+T)}(1-2\lambda T)^{-1/2}. \tag{36}$$

Take $\theta, \gamma\epsilon(0,1)$ and set all $K\epsilon N$

$$\beta_k = (1+\gamma)^2 h(\theta^k)$$

and

$$\lambda_k = [2\theta^k(1+\gamma)]^{-1}.$$

Applying (22), we see that for all $K\epsilon N$,

$$P\left[\max_{0\leq t\leq\theta^k} 2X_t^d \geq (1+\gamma)^2 h(\theta^k) \leq e^{-1/2(1+r)}(1+r^{-1})(-klog\theta)^{-(1+\gamma)}\right].$$

It follows from the Borel-Cantelli lemma that for almost all $w\epsilon\Omega$ and since

$$\sum_{k>0} \frac{1}{K^{(1+\gamma)}},$$

there exists a natural number $K^{\theta,\gamma}(w)$ such that for all $k \geq K^{\theta,\gamma}(w)$,

$$\max_{0\leq t\leq\theta^k} 2X_t^d(w) < (1+\gamma)^2 h(\theta^k).$$

In particular, for all $t \in (\theta^{k+1}, \theta^k)$,

$$2X_t^d(w) < (1+\gamma)^2 h(\theta^k) \leq (1+\gamma)^2 \frac{h(t)}{\theta}.$$

Hence

$$\lim_{t\to 0} \sup \frac{X_t^d}{h(t)} \leq \frac{(1+\gamma)^2}{2\theta} \text{a.s.}$$

2.3 The Equilibrium Price and Growth Rate

2.3.1 The particular case

When the mean interest rate of some stocks does not depend on other stocks in the market, we consider again the following stochastic differential equation (SDE);

$$dP_t = P_t(\mu dt + \sigma(M_t)dW_t), \quad P_0 = \varepsilon_0. \tag{37}$$

Equation (35) is our so called multifractal spectrum model defined in continuous time. The most probable path $\varphi(t)$ associated with this equation satisfies

$$\varphi^{'}(t) = \left(\mu - \frac{1}{2}\sigma^2(M_t)\right)\varphi, \; \varphi(0) = \varepsilon. \tag{38}$$

Equation (36) means that an investor has invested his money in a stock with a linear mean return μ_t and volatility $\sigma(M_t)$ and his real return rate is most likely to be given by

$$c(t) = \mu - \frac{1}{2}\sigma^2(M_t), \tag{39}$$

Instead of the usual μ.

The Ito's Formula on (35) leads to the model equation

$$-\frac{\partial W}{\partial t} = \mu P \frac{\partial W}{\partial P} + \frac{1}{2}\sigma^2(M_t)P^2\frac{\partial^2 W}{\partial P^2} - rW, t \geq t_0, P_0 = \varepsilon_0, \tag{40}$$

where $W = W(P_t, K_t)$ is the investment worth and K_t is the investment over period $t.$ Following the assumption in [22] and the references therein, we have (40) reduced to an ordinary differential equation of the form

$$\mu P \frac{dW}{dP} + \frac{1}{2}\sigma^2(M_t)P^2\frac{d^2W}{dP^2} - rW = -P. \tag{41}$$

The solution of (39) by the method of change of variable using Euler's substitution is;

$$W(P) = AP^{\lambda_1} + \frac{P}{r-\mu}, \tag{42}$$

with

$$A\lambda_1\hat{P}^{\lambda_1} + \frac{\hat{P}}{r-\mu} = 0, \tag{43}$$

where

$$\lambda_1 = \frac{-\left(\mu - \frac{1}{2}\sigma^2(M_t)\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2(M_t)\right)^2 + 2\sigma(M_t)r}}{\sigma^2(M_t)}, \tag{44}$$

is the positive characteristics root of (41).

We have assumed here that $W(P)$ is twice differentiable such that

$$W(0) = 0 \text{ and } \frac{dW(P)}{dP} = 0. \tag{45}$$

Using equations (7), (15), (19) and (21), we have (40) as

$$W(P) = AP^{\lambda_1} + \frac{(\Delta t)^{-(\tau(q)+1)}P}{(\alpha_0 + \sum_{k=1}^{d} \alpha_{k+1}(P_{t-k} - \mu_{t-k})^2)}, \tag{46}$$

with (using 45b)

$$A\lambda_1 \hat{P}^{\lambda_1} - \frac{\mu(\Delta t)^{-(\tau(q)+1)}}{\sigma(\hat{P}_u - \mu_u)^2} = 0\ . \tag{47}$$

Equation (46) is the worth growth rate of an investor under MSM.

Under equilibrium condition, the discounted profit from a unit capacity at $\hat{P}$ must be equal to the expected unit cost $\bar{P}$ of the risky stock option. This therefore implies that by (47), we have

$$W(P) = A\lambda_1 \hat{P}^{\lambda_1} - \frac{\mu(\Delta t)^{-(\tau(q)+1)}}{\sigma(\hat{P}_u - \mu_u)^2} = \bar{P}. \tag{48}$$

Solving for A in (47) and (48) and equating the results gives;

$$\hat{P} = \frac{\lambda_1 \sigma(\Delta t)^{(\tau(q)+1)}\mu\bar{P}}{[(\lambda_1 - 1)(\sigma(\Delta t)^{(\tau(q)+1)}\bar{P}) - 1]}. \tag{49}$$

2.3.2 The general case

In the real world in general, markets are neither ideal nor complete. Therefore the model equation (40) can hardly be seen as real world behaviour of a stock market price.

Consider a market comprising h unit of asset in long position and one unit of the option in short position. At time T the market value is assumed to be $h - p(E)$ or $h(s) - v$.

After an elapse,Δ_t, the value of the market will change by an amount/rate $h(\Delta P + \vartheta \Delta t) - \Delta W$,in view of the dividend received on h unit held, where ϑ is the market price of risk. By Ito's lemma we have

$$h(\mu P\Delta t + \sigma P\Delta t + \vartheta\Delta t) = \left[\left(\frac{\partial W}{\partial t} + \frac{\partial W}{\partial P}\mu P + \frac{1}{2}\frac{\partial^2 W}{\partial P^2}\sigma^2 P^2\right)\Delta t + \frac{\partial W}{\partial P}\sigma P\Delta z\right],$$

or

$$\left[h\mu P + h\vartheta - \left(\frac{\partial W}{\partial t} + \frac{\partial W}{\partial P}\mu P + \frac{1}{2}\frac{\partial^2 W}{\partial P^2}\sigma^2 P^2\right)\right]\Delta t = \left[h\sigma P - \frac{\partial W}{\partial P}\sigma P\right]\Delta t. \tag{50}$$

Take $h = \frac{\partial W}{\partial P}$, then the uncertainty term disappears and the market in this case is temporarily riskless (no signal). It should therefore grow in value by the riskless rate in force i.e.

$$\left(h\mu P + h\vartheta - \left[\frac{\partial W}{\partial t} + \frac{\partial W}{\partial P}\mu P + \frac{1}{2}\frac{\partial^2 W}{\partial P^2}\sigma^2 P^2\right]\right)\Delta t = (hP - W)r\Delta t. \tag{51}$$

Thus,

$$\vartheta\frac{\partial W}{\partial P} - \left(\frac{\partial W}{\partial t} + \frac{1}{2}\frac{\partial^2 W}{\partial P^2}\sigma^2 P^2\right) = \left(\frac{\partial W}{\partial P}P - W\right)r, \tag{52}$$

so that

$$\frac{\partial W}{\partial t} + (rP - \vartheta)\frac{\partial W}{\partial P} + \frac{1}{2}\frac{\partial^2 W}{\partial P^2}\sigma^2 P^2 = rW\ . \tag{53}$$

Under the following dynamics

$$dP_t = \alpha(t)P_t dt + \sigma(M_t)P_t dW(t),$$

we have the MSM version of the parabolic partial differential equation (53) with $\vartheta = 0$ as;

$$-\frac{\partial W(P,t)}{\partial t} = rP\frac{\partial W(P,t)}{\partial P} + \frac{1}{2}\sigma^2(M_t)P^2\frac{\partial^2 W(P,t)}{\partial P^2} - rW(P,t), \forall (P,t) \in (0,\infty)\times(0.T), \tag{54}$$

where $\alpha(t) = \ln\left(\frac{P_{t+\Delta t} - P_t}{\Delta t}\right)$ is the rate of stock price changes at time t.

Theorem 2

Let $W(P,t)$ be the investment output, r the linear discount rate (as in equation (7)) and P the stock prices. The PDE

$$\frac{1}{2}\sigma^2(M_t)P^2\frac{\partial^2 W(P,t)}{\partial P^2} + r\frac{P\partial W(P,t)}{\partial P} - rW(P,t) = \frac{\partial W(P,t)}{\partial t}$$

with

$$\bar{W}(\bar{P},t) = \bar{W}_0, \tag{55}$$

has solution of the type

$$\begin{aligned}\bar{W}(\bar{P},t) &= \bar{W}(0)\exp\left[-\left\{\frac{2\,ln\left(\frac{P_{t+Dt}-P_t}{Dt}\right)\bar{P}^{-2}}{(\Delta t)^{\tau(q)+1}h(\xi_{t-1},\xi_{t-2},\dots,\xi_{t-d},\alpha)\bar{P}_u - \mu_u}\right.\right.\\ &\left.\left.+\frac{\left(\mu-\frac{1}{2}\sigma^2(M_t)\right)+\sqrt{\left(\mu-\frac{1}{2}\sigma^2(M_t)\right)^2+2\sigma(M_t)r}}{\sigma^2(M_t)}\bar{P}\right\}\right]\\ &= \bar{W}(0)exp\left[-\left\{\frac{2\,ln\left(\frac{P_{t+\Delta t}-P_t}{\Delta t}\right)\bar{P}^{-2}+\sqrt{\left(\mu-\frac{1}{2}\sigma^2(M_t)\right)^2+2\sigma(M_t)r}}{(\Delta t)^{\tau(q)+1}\left(\alpha_0+\sum_{k=1}^{d}\alpha_t\varepsilon_{t-k}\right)}\right\}\right].\end{aligned} \tag{56}$$

Furthermore,

$$\frac{\partial \bar{W}(\bar{P},t)}{\partial \bar{p}} = 0\,, \forall t, \tag{57}$$

with

$$\left(\mu - \frac{1}{2}\sigma^2(M_t)\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2(M_t)\right)^2 + 2\sigma(M_t)r} + \frac{4\,ln\left(\frac{P_{t+Dt}-P_t}{Dt}\right)}{p^3(\Delta t)^{\tau(q)+1}h(\xi_{t-1},\xi_{t-2},\dots,\xi_{t-d},\alpha)\hat{P}_u - \mu_u} = 0. \tag{58}$$

Proof:

To remove r from (54), set

$$\bar{W} = e^{-rt}W \tag{59}$$

and

$$\bar{P} = e^{-rt}P, \tag{60}$$

so that (54) becomes;

$$\frac{1}{2}\sigma^2(M_t)\bar{P}^2\frac{\partial^2\bar{W}(P,t)}{\partial\bar{P}^2} + \frac{\partial\bar{W}(P,t)}{\partial t} = 0. \tag{61}$$

By the method of separation we obtain [34], we obtain a special solution of the form;

$$\bar{W} = W_0 \exp\left\{\frac{-2\alpha t\bar{P}^{-2}}{\sigma^2(M_t)} + \lambda\bar{P}\right\}. \tag{62}$$

Using equations (7), (18) and (19) into (62), we have easily, (56).

Again, solving for $\bar{P}$ in equation (60) and equating to (58) gives (using (20) and if λ is as in (44));

$$\bar{P} = \left[\frac{4ln(P_t - P_{t+\Delta t})(\Delta t)^{-T(q)}}{\left(\left(\mu-\frac{1}{2}\sigma^2(M_t)\right)+\sqrt{\left(\mu-\frac{1}{2}\sigma^2(M_t)\right)^2+2\sigma(M_t)r}\right)\left(\alpha_0+\sum_{k=1}^{d}\alpha_t\varepsilon_{t-k}\right)}\right]^{1/3}. \tag{63}$$

Also equating equations (63) and (59) gives

$$P = \left[\frac{4ln(P_t - P_{t+\Delta t})(\Delta t)^{-T(q)}}{\lambda\left(\left(\alpha_0 + \sum_{k=1}^{d}\alpha_k\xi_{t-k}\right)\right)}\right]^{1/3} e^{(\mu+\sigma/m_t)\xi_t)}$$

$$= \left[\frac{4ln(P_t - P_{t+\Delta t})(\Delta t)^{-T(q)}}{\lambda\left(\left(\alpha_0 + \sum_{k=1}^{d}\alpha_k\xi_{t-k}\right)\right)}\right]^{1/3} \exp\left\{\mu + (\Delta t)^{-(T(q)+1)}\left(\alpha_0 + \sum_{k=1}^{d}\alpha_k\xi_{t-k}\right)P_t - \mu_t\right\},$$

(using equations (7), (18) and (19)). We therefore have the future price as

$$P_t = \left[\left[\frac{4ln(P_t - P_{t+\Delta t})(\Delta t)^{-T(q)}}{\lambda\left(\left(\alpha_0 + \sum_{k=1}^{d} \alpha_k \xi_{t-k}\right)\right)}\right]^{1/3}\right] exp\{\mu + (\Delta t)^{-(T(q)+1)}\left(\alpha_0 + \sum_{k=0}^{d} \alpha_{k+1}(P_t - \mu_t)^2\right\}. \quad (64)$$

In general we have the growth rate of the investor's portfolio as (using equations (59) and (56))

$$W = \bar{W}(0)exp\left[-\left\{\frac{2\,ln\left(\frac{P_{t+\Delta t} - P_t}{\Delta t}\right)\bar{P}^{-2} + \sqrt{\left(\mu - \frac{1}{2}\sigma^2(M_t)\right)^2 + 2\sigma(M_t)r}}{(\Delta t)^{\tau(q)+1}\left(\alpha_0 + \sum_{k=1}^{d} \alpha_t \varepsilon_{t-k}\right)}\right\}\right] exp\left\{\left(\mu + \bar{\sigma}\left(M_{1,t}M_{2,t} \ldots M_{\bar{k},t}\right)^{1/2}\xi_t\right)t\right\}. \quad (65)$$

2.4 The Optimal Trading Strategy

Let us denote an optimal trading strategy π_t^* for which we define [35]

$$H_{\pi_t^*}(t, r, P) = E_{\pi_t^*}[U(W(T))]. \quad (66)$$

Our objective here is to find the optimal value function such that

$$H(t, r, P) = \mathrm{Sup}_{\pi_t^* \in \pi} H_{\pi_t^*}(t, r, P). \quad (67)$$

Assume equation (54) for $\vartheta \neq 0$ together with the optimal π_t^* , we have

$$\frac{\partial W}{\partial t} + (rP - \vartheta)\frac{\partial W}{\partial P} + \frac{1}{2}\frac{\partial^2 W}{\partial P^2}\sigma^2(M_t)\pi_t^* P^2 = rW. \quad (68)$$

Put $z = \frac{\alpha}{P}$; $W(P) = z^\beta H(Z)$. Since W is not dependent on r, differentiating and substituting into (68), gives

$$rz^\beta w = \frac{\sigma^2(M_t)}{2}\left[\beta(\beta+1)z^\beta w + \beta z^{\beta+1}\frac{\partial H}{\partial z} + 2(\beta+1)z^{\beta+1}\frac{\partial H}{\partial z} + z^{\beta+2}\pi_t^*\frac{\partial^2 H}{\partial z^2}\right]$$

$$+\left(\frac{r\alpha}{z} - \vartheta\right)\left(-\frac{1}{\alpha}\right)\left(\beta z^{\beta+1}w + z^{\beta+2}\frac{\partial w}{\partial z}\right). \quad (69)$$

Now cancelling by z^β and collecting like terms gives

$$0 = \frac{\sigma^2(M_t)\pi_t^*}{2}z^2\frac{\partial^2 H}{\partial z^2} + \frac{\partial H}{\partial z}z\left(\sigma^2(\beta+1) - r - \frac{D}{\alpha}z\right) + H\left(\frac{\sigma^2}{2}\beta(\beta+1) - r(\beta+1) + \beta\frac{D}{\alpha}z\right). \quad (70)$$

Set $\frac{\sigma^2(M_t)}{2}\beta = r$ and let $\frac{\vartheta/\alpha}{\frac{\sigma^2(M_t)}{2}} = -1$ to have $\beta = \frac{2r}{\sigma^2(M_t)}, \alpha = -\frac{2D}{\sigma^2(M_t)}$. Substitute into (70) to obtain

$$z\pi_t^* H_{zz} + (2 - z)H_z = \beta(H - H_z). \quad (71)$$

Solving for optimal trading strategy, we have

$$\pi_t^* = \frac{\beta(H - H_z)}{zH_{zz}} + \frac{(z-2)H_z}{zH_{zz}}$$

$$= \frac{\frac{2r}{\sigma^2(M_t)}(H - H_z)}{zH_{zz}} + \frac{(z-2)H_z}{zH_{zz}}$$

$$= \frac{\frac{2\mu + \sigma(M_t)P_t - \mu_t,}{(\Delta t)^{\tau(q)+1}\left(\alpha_0 + \sum_{k=1}^{d} \alpha_k \left(P_{t-k} - \mu_{t-k}\right)^2\right)}(H - H_z)}{zH_{zz}} + \frac{(z-2)H_z}{zH_{zz}}. \quad (72)$$

3 Conclusion

The multi-fractal spectrum model has some successes in explaining excess volatility of stock returns compared to fundamentals and negative Skewness of equity returns as well as generating multi-fractal spectrum. By (50) there is no market signal as it tends to zero, meaning that the market is likely to crash at such point, signifying insolvency in asset returns. Equation (23a) can be represented by a renormalized probability distribution of local Holder exponents, called the multifractal spectrum by the form

$$f(\alpha) = \lim_{k \to \infty} \left(\frac{\ln N(\alpha)}{\ln s^k}\right). \quad (73a)$$

Denoting by $N(\alpha, \Delta t)$ the number of intervals $[t, t + \Delta t]$ required to cover $\tau(\alpha)$ we can write (23b)

$$N(\alpha, \Delta t) \sim (\Delta t)^{-f(\alpha)}. \quad (73b)$$

The market prices correspond to the values of α between α_{min} and $\alpha(f_{max})$ [36].

Furthermore, equation (71) can be solved for the value function H and replaced in (72) in order to obtain the optimal investment strategies.

We have only consider herein, the continuous case. An on-going work considers an extension where we will derive the MSM with the discrete version and establish its comparative effectiveness with the continuous case. An empirical illustration to compare both the MSM for equity returns and those in existing literature will be shown.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Frisch U. Turbulence. Cambridge University Press, Cambridge; 1995.

[2] Mandelbrot BB. Intermittent turbulence in self-similar cascades: Divergence of high moments and dimension of the carrier. J Fluid Mech. 1974;62:331-358.

[3] Meneveau C, Sreenivasan KR. The multifractal nature of turbulent energy-dissipation. J Fluid Mech. 1991;224:429-484.

[4] Calvet L, Fisher A. Multifractal Volatility Theory, forecasting and pricing. Burlington, MA: Academic Press; 2008.

[5] Taylor SJ. Modeling financial time series (2nd ed.). New Jersey: World Scientist; 2008.

[6] Wiggins JB. Option values under stochastic volatility: Theory and empirical estimates. Journal of Financial Economics. 1987;19(2):351-372.

[7] Hamilton JD. A New Approach to the Economics Analysis of Non stationary Time Series and the Business Cycle. Econometric a (The Econometric Society). 2008;57(2):357-384.

[8] Mandelbrot BB, Fisher A, Calvet L. A multifractal model of asset returns. Discussion Paper, Cowles Foundation Yale University. 1997;1164-1166.

[9] Mandelbrot BB. Intermittent turbulence in self- similar cascades: Divergence of high moments and dimension of the carrier". Journal of Fluid Mechanics. 2006;62(2):331-356.

[10] Mandelbrot BB. The fractal geometry of nature (Updated and augm. ed.). New York: Freeman; 1983.

[11] Mandelbrot BB, Berger JM. Multifractals and 1/f noise: Wild self-affinity in physics (1963-1976). (Repr. ed.). New York, NY [u.a.]: Springer; 1999.

[12] Calvet L, Fisher A, Thompson SB. Volatility Co-movement: A multifrequency approach. Journal of Econometrics. 2006;131:179.

[13] Calvet L. How to forecast Long-Run Volatility: Regime switching and the estimation of multifractal processes. Journal of Financial Econometrics. 2004;2:49-83.

[14] Gray SF. Modeling the conditional distribution of interest rates as a regime-switching process. Journal of financial economics. 1996;42:27-77.

[15] Klaassen F. Improving GARCH volatility forecasts with regime-switching GARCH. Empirical Economics. 2002;27(2):363-394.

[16] Bollerslev TO, Mikkelsen H. Modeling and pricing long memory in stock market volatility. Journal of Econometrics. 1996;73:151-162.

[17] Lux T. The Markov-Switching Multifractal Model of Asset Returns. Journal of Business & Economic Statistics. 2008;26(2):194-210.

[18] Black F, Scholes M. The valuation of options and corporate liabilities. J Pol Econ. 1973;81:637-654.

[19] Black F, Karasinski P. Bond and options pricing when short rate and lognormal. Finan Analysis J. 1991;47(4):52-59.

[20] Hull J, White A. The pricing of options on Assets with stochastic Volatilities. J Finan. 1987;42:271-301.

[21] Ugbebor OO, Onah SE, Ojowu O. An empirical stochastic model of stock price changes. J Nig Math Soc. 2001;20:95-101.

[22] Osu BO, Okoroafor AC. On the measurement of random behaviour of stock price changes. J Math Sci Dattapukur. 2007;18(2):131-141. STMAZ 05343605, MR2388931.

[23] Osu BO, Okoroafor AC, Olunkwa C. Stability analysis of stochastic model of stock market price. Afri J Math Comp Sci Res. 2009;2(6):098-103.

[24] Eberlein E, Keller U. Hyperbolic distributions in finance. Bernoulli. 1995;1(3):281-299.

[25] Calvet L, Fisher A. Regime-switching and the estimation of multifractal processes (Technical report). Working paper series. National Bureau of Economic Research. 9839; 2003.

[26] Hausdorff F. Dimension Und 8ussees mass Mathematische Annalen. 1919;79:157-179.

[27] Calvet L, Fisher A. Forecasting multifractal volatility. Journal of Econometrics. 2001;105:27.

[28] Taylor SJ. The Hausdorff $\alpha -$ dimensional of Brownian paths in n-space. Proc Cambridge Philos Soc. 1953;49:31-39.

[29] Taylor SJ. On the connection between generalized capacities and Hausdorff measure. Proc Cambridge Philos Soc. 1961;57:524-531.

[30] Jafford S. The multifractal nature of Levy processes. Probab Theory and Rel Fields. 1999;114:207-229.

[31] Xiao L. Random fractals and Markov processes in fractal geometry and applications. Ame Mat Soc. 2004;261-338.

[32] Uzoma PU. An estimate of the exact size of the singular points of Laplace equation in Euclidean space (a case study of the earthling screen). An M.Sc. dissertation in mathematics department. Abia State University, Uturu, Nigeria; 2006.

[33] Arneodo A, Bacry E, Muzy JF. The thermodynamics of fractals revisited with wavelets. Physica A. 1995;213:232-275.

[34] Osu BO, Ihedioha SA, Ekuma-Okereke E. The Price of Portfolio Selection under Tail Conditional Expectation with Consumption cost and Transaction Cost. Afrika Statistika. 2013;8:545-559. DOI: http://dx.doi.org/10.4314/afst.v8i1.4 Zbl 1279.91148.

[35] Zhang C, Rong X. Optimal investment strategy for DC pension with stochastic salary under the affine interest rate model. Discrete Dynamics in nature and Society. Hindawi Publishing Corporation; 2013. Article ID 297875, 11 pages. http://dx.doi./10.1155/2013/297875.

[36] Piacquadio M, Redelico FO. Multifractal analysis and instability index of prior-to-crash market situations; 2009. A$arXiv$: $0910.2474\ v1[q.fin.ST]$.

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