British Journal of Mathematics & Computer Science
9(2): 160-173, 2015, Article no.BJMCS.2015.194
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

SCIENCEDOMAIN

Solutions of Some Minimization Problems of a Special Class of Functions

S. T. Obeidat[1*] and M. Motawi Khashan[2]

[1] *Department of Mathematics, Preparatory Year, University of Hail, Kingdom of Saudi Arabia.*
[2] *Department of Mathematics, King Saud University, Riyadh, Kingdom of Saudi Arabia.*

Article Information

DOI: 10.9734/BJMCS/2015/17254
Editor(s):
(1) Sheng Zhang, Department of Mathematics, Bohai University, Jinzhou, China.
Reviewers:
(1) Lakshmi Narayan Mishra, Department of Mathematics, National Institute of Technology, Silchar, India.
(2) Anonymous, China.
Complete Peer review History:
http://www.sciencedomain.org/review-history.php?iid=1143&id=6&aid=9338

Original Research Article

Received: 06 March 2015
Accepted: 07 April 2015
Published: 21 May 2015

Abstract

Solutions of three minimization problems of some classes of functions are obtained. The findings of the present paper generalize several results related to quantile regression. The new results can be applied in different aspects such as weighted l_1-norm minimization of curves in $\mathbb{R}^d$, where d is a positive integer, and approximation theory. Some applications of the new results are introduced.

Keywords: Global optimization; minimization; l_1-norm.

2010 Mathematics Subject Classification: 90C26, 90C90, 90-08

*Corresponding author: E-mail: s.obeidat@uoh.edu.sa;

1 Introduction

In the present paper we consider the following two minimization problems:

Problem 1:

$$\begin{cases} \text{Minimize } \sum_{r=1}^{n} T_{s,b}(x-a_r), x \in \mathbb{R}, \\ \text{where } T_{s,b}(x) = \begin{cases} bx & \text{If } x \geqslant 0 \\ -sx & \text{If } x < 0 \end{cases}, \\ b, s > 0, \text{ and } a_r < a_{r+1} \text{ for all } 1 \leq r \leq n-1 \end{cases}.$$

Problem 2:

$$\begin{cases} \text{Minimize } \sum_{r=1}^{n} c_r |x-a_r|, \ x \in \mathbb{R}, \\ \text{where } \{c_r\}_{r=1}^{\infty} \text{ is a sequence of positive real numbers,} \\ \text{and } a_r < a_{r+1} \text{ for all } 1 \leq r \leq n-1 \end{cases}.$$

In fact, these problems generalize the following problem:

Problem 0:

$$\begin{cases} \text{Minimize } \sum_{r=1}^{n} |x-a_r|, x \in \mathbb{R}, \\ \text{where } a_r < a_{r+1}, \text{ for all } 1 \leq r \leq n-1 \end{cases},$$

which was discussed in several papers under some constraints, see for example [1], [2] and [3]. In addition, a generalization of this problem was obtained in [4].

The importance of the above mentioned problems is due to their applications in several fields, e.g., in regression and approximation theory. In addition, Problem 1 is equivalent to the following separable constrained problem:

Problem 3:

$$\begin{cases} \text{Minimize } \sum_{r=1}^{n} T_{s,b}(x_r-a_r), \\ \text{Subject to } \begin{cases} (x_1-x_2) + \sum_{i\neq 1,2} (x_1-x_i)^2 \leq 0 \\ (x_2-x_3) + \sum_{i\neq 2,3} (x_2-x_i)^2 \leq 0 \\ (x_3-x_4) + \sum_{i\neq 3,4} (x_3-x_i)^2 \leq 0 \\ \vdots \\ (x_{n-1}-x_n) + \sum_{i\neq n-1,n} (x_{n-1}-x_i)^2 \leq 0 \\ (x_n-x_1) + \sum_{i\neq 1,n} (x_n-x_i)^2 \leq 0 \end{cases} \\ \text{where } b, s > 0, \ a_r < a_{r+1}, \text{ for all } 1 \leq r \leq n-1, \text{ and } x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n. \end{cases},$$

and, Problem 2 is equivalent to the following separable constrained problem:

Problem 4:

$$\begin{cases} \text{Minimize } \sum\limits_{r=1}^{n} c_r \left| x_r - a_r \right|, \\ \text{Subject to } \begin{cases} (x_1 - x_2) + \sum\limits_{i \neq 1,2} (x_1 - x_i)^2 \leq 0 \\ (x_2 - x_3) + \sum\limits_{i \neq 2,3} (x_2 - x_i)^2 \leq 0 \\ (x_3 - x_4) + \sum\limits_{i \neq 3,4} (x_3 - x_i)^2 \leq 0 \\ \cdot \\ \cdot \\ \cdot \\ (x_{n-1} - x_n) + \sum\limits_{i \neq n-1,n} (x_{n-1} - x_i)^2 \leq 0 \\ (x_n - x_1) + \sum\limits_{i \neq 1,n} (x_n - x_i)^2 \leq 0 \end{cases} . \\ \text{where } \{c_r\}_{r=1}^{\infty} \text{ is a sequence of positive numbers,} \\ a_r < a_{r+1}, \text{ for all } 1 \leq r \leq n-1, \text{ and } x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n \end{cases}$$

Separable problems arise frequently in practice, particularly for time-dependent optimization, and hence solving separable problems is important for their applicability in different aspects. Different techniques for solving separable problems were obtained in different articles, see for example [5] and [6].

In the present paper, we solve Problem 1 and Problem 2. Then we obtain more general forms of them, and obtain some corollaries. Next, we discuss some applications of the obtained results in different contexts. The techniques developed in this paper can be applied to generalize the results obtained in [7] and [8], which will be our target in future work.

Throughout this paper, $T_{s,b}$ and $L_{s,b,A}$, where s and b are positive integers and $A = \{a_r\}_{r=1}^{n}$ is a strictly increasing sequence of real numbers,denote the functions which are defined over $\mathbb{R}$, as follows:

$$T_{s,b}(x) = \begin{cases} bx & \text{If } x \geqslant 0 \\ -sx & \text{If } x < 0 \end{cases},$$

and

$$L_{s,b,A}(x) = \sum_{r=1}^{n} T_{s,b}(x - a_r).$$

2 Main Results

Theorem 2.1. *Let s, b be positive numbers, and let $A = \{a_r\}_{r=1}^{n}$ be a strictly increasing sequence of real numbers, where $n > 1$ is an integer.*

Then

1. *If $[kb - (n-k)s] = 0$ for some $1 \leq k < n$, $L_{s,b,A}$ attains its smallest value at any value $t \in [a_k, a_{k+1}]$.*
2. *If $b - (n-1)s > 0$, $L_{s,b,A}$ attains its smallest value at a_1.*
3. *If $(n-1)b - s < 0$, $L_{s,b,A}$ attains its smallest value at a_n.*

4. *If* $[(k-1)b-(n-k+1)s][kb-(n-k)s] < 0$ *for some* $1 < k < n$, $L_{s,b,A}$ *attains its smallest value at* a_k

Proof. First of all, for $x \leq a_1$,

$$L_{s,b,A}(x) = -s\sum_{r=1}^{n}(x-a_r) = -snx + s\sum_{r=1}^{n}a_r \ ,$$

and for $a_{k-1} \leq x \leq a_k$,where $1 < k < n$,

$$L_{s,b,A}(x) = b\sum_{r=1}^{k-1}(x-a_r) - s\sum_{r=k}^{n}(x-a_r) = [(k-1)b-(n-k+1)s]x + M_k,$$

where

$$M_k = s\sum_{r=k}^{n}a_r - b\sum_{r=1}^{k-1}a_r.$$

Also, for $x \geq a_n$,

$$L_{s,b}(x) = b\sum_{r=1}^{n}(x-a_r) = bnx - b\sum_{r=1}^{n}a_r \ .$$

Clearly, $L_{s,b,A}$ is decreasing over $(-\infty, a_1]$ and is increasing over $[a_n, \infty)$. In addition, for each $1 < k < n$, we have

$$[(k-1)b-(n-k+1)s] < [kb-(n-k)s].$$

Now, we consider the following cases:

Case 2.2. $[kb-(n-k)s] = 0$ *for some* $1 \leq k < n$.

If $k = 1$ *then*$L_{s,b,A}$ *is constant over* $[a_1, a_2]$. *Since* $0 = b-(n-1)s < ib-(n-i)s$ *for each* $1 < i < n$, *and since* $L_{s,b,A}$ *is increasing over* $[a_n, \infty)$ *,we conclude that* $L_{s,b,A}$ *is increasing over* $[a_2, \infty)$. *Since* $L_{s,b,A}$ *is decreasing over* $(-\infty, a_1]$, $L_{s,b,A}$ *attains its smallest value at any* $t \in [a_1, a_2]$.

If $k = n-1$ *then* $L_{s,b,A}$ *is constant over* $[a_{n-1}, a_n]$. *Since* $ib-(n-i)s < (n-1)b-s = 0$ *for each* $1 \leq i < n-1, L_{s,b,A}$ *is decreasing over* $[a_1, a_{n-1}]$. *Since* $L_{s,b,A}$ *is decreasing over* $(-\infty, a_1]$ *and is increasing over* $[a_n, \infty)$, $L_{s,b,A}$ *attains its smallest value at any* $t \in [a_{n-1}, a_n]$.

If $1 < k < n-1$ *then* $L_{s,b,A}$ *is constant over* $[a_k, a_{k+1}]$. *Since* $ib-(n-i)s < kb-(n-k)s = 0 < jb-(n-j)s$ *for each* $1 \leq i < k$ *and* $k < j < n, L_{s,b,A}$ *is decreasing over* $[a_1, a_k]$ *and is increasing over* $[a_{k+1}, a_n]$. *Since* $L_{s,b,A}$ *is decreasing over* $(-\infty, a_1]$ *and is increasing over* $[a_n, \infty)$, $L_{s,b,A}$ *attains its smallest value at any* $t \in [a_k, a_{k+1}]$.

Case 2.3. $b-(n-1)s > 0$.

Since $b-(n-1)s < ib-(n-i)s$ *for each* $1 < i < n, L_{s,b}$ *is increasing over* $[a_1, a_n]$. *Since* $L_{s,b,A}$ *is decreasing over* $(-\infty, a_1]$ *and is increasing over* $[a_n, \infty)$, $L_{s,b,A}$ *attains its smallest value at* a_1.

Case 2.4. $(n-1)b-s < 0$.

Since $(n-1)b-s > ib-(n-i)s$ *for each* $1 \leq i < n-1, L_{s,b}$ *is decreasing over* $[a_1, a_n]$. *Since* $L_{s,b,A}$ *is decreasing over* $(-\infty, a_1]$ *and is increasing over* $[a_n, \infty)$, $L_{s,b,A}$ *attains its smallest value at* a_n.

Case 2.5. $[(k-1)b-(n-k+1)s][kb-(n-k)s] < 0$ *for some* $1 < k < n$.

Since $[(k-1)b-(n-k+1)s] < [kb-(n-k)s]$, *we conclude that*

$$[(k-1)b-(n-k+1)s] < 0 \ \textit{and} \ [kb-(n-k)s] > 0.$$

Also, since $kb-(n-k)s < ib-(n-i)s$ *for each* $k < i < n$ *and* $(k-1)b-(n-k+1)s > jb-(n-j)s$ *for each* $1 \le j < k-1$, $L_{s,b,A}$ *is decreasing over* $[a_1, a_k]$ *and is increasing over* $[a_k, a_n]$. *Since* $L_{s,b,A}$ *is decreasing over* $(-\infty, a_1]$ *and is increasing over* $[a_n, \infty)$, $L_{s,b,A}$ *attains its smallest value at* a_k.

□

Corollary 2.6. *Let* $q > 0$ *and* $A = \{a_r\}_{r=1}^n$ *be a strictly increasing sequence of real numbers, where* $n > 1$ *is an integer. Define* $g : \mathbb{R} \longrightarrow [0, \infty)$ *as:*

$$g(x) = \sum_{r=1}^{n} q\,|x - a_r|, x \in \mathbb{R}.$$

Then

1. *If* n *is odd,* g *attains its smallest value at* $a_{\frac{n+1}{2}}$.
2. *If* n *is even,* g *attains its smallest value at any* $x \in \left[a_{\frac{n}{2}}, a_{\frac{n+2}{2}}\right]$.

Proof. Note that g is nothing but $L_{s,b,A}$ in Theorem 2.1 with $s = b = q$. If n is odd then

$$\left[\left(\frac{n+1}{2} - 1\right)b - \left(n - \frac{n+1}{2} + 1\right)s\right]\left[\frac{n+1}{2}b - \left(n - \frac{n+1}{2}\right)s\right] = (-q)(q) < 0,$$

which implies by Theorem 2.1 that g attains its smallest value at $a_{\frac{n+1}{2}}$. If n is even then

$$\frac{n}{2}b - \left(n - \frac{n}{2}\right)s = 0,$$

which implies by Theorem 2.1 that g attains its smallest value at any $x \in \left[a_{\frac{n}{2}}, a_{\frac{n+2}{2}}\right]$. □

The next corollary shows a relationship between $L_{s,b,A}$ and $L_{b,s,A}$ regarding the fact where they attain their smallest values.

Corollary 2.7. *Let* s, b *be positive numbers, and let* $\{a_r\}_{r=1}^n$ *be a strictly increasing sequence of real numbers, where* $n > 1$ *is an integer. Then*

1. *If* $L_{s,b,A}$ *attains its smallest value at* a_i, *for some* $1 \le i \le n$, *then* $L_{b,s,A}$ *attains its smallest value at* a_{n-i+1}.
2. *If* $L_{s,b,A}$ *attains its smallest value at any* $t \in [a_k, a_{k+1}]$, $1 \le k < n$, *then* $L_{b,s,A}$ *attains its smallest value at any* $t \in [a_{n-k}, a_{n-k+1}]$.

Proof. 1. If $L_{s,b,A}$ attains its smallest value at a_1, then $b-(n-1)s > 0$ which implies that $(n-1)s - b < 0$. By Theorem 2.1 (Part 3), $L_{b,s,A}$ attains its smallest value at a_n.

If $L_{s,b,A}$ attains its smallest value at a_n, then $(n-1)b - s < 0$ which implies that $s - (n-1)b > 0$. By Theorem 2.1 (Part 2), $L_{b,s,A}$ attains its smallest value at a_1.

$L_{s,b,A}$ attains its smallest value at a_k for some $1 < k < n$, then

$$[(k-1)b-(n-k+1)s][kb-(n-k)s] < 0.$$

But

$$\begin{aligned}
& [(k-1)b-(n-k+1)s][kb-(n-k)s] \\
= \; & [kb-(n-k)s][(k-1)b-(n-k+1)s] \\
= \; & [(n-k)s-kb][(n-k+1)s-(k-1)b] \\
= \; & [((n-k+1)-1)s-(n-(n-k+1)+1)b][(n-k+1)s-(n-(n-k+1))b],
\end{aligned}$$

which implies by Theorem 2.1 (Part 4) that $L_{b,s,A}$ attains its smallest value at a_{n-k+1}.

2. If $L_{s,b,A}$ attains its smallest value at any $t \in [a_k, a_{k+1}]$, $1 \le k < n$, then $[kb - (n-k)s] = 0$ which implies that $[(n-k)s - kb] = 0$. But
$$[(n-k)s - kb] = [(n-k)s - (n-(n-k))b].$$
Thus, by Theorem 2.1 (Part 1), $L_{b,s,A}$ attains its smallest value at any $t \in [a_{n-k}, a_{n-k+1}]$. □

Corollary 2.8. *Let $t \in \mathbb{R}$ and $\{a_r\}_{r=1}^{n}$ be a strictly increasing sequence of real numbers, where $n > 1$ is an integer. Define $g : \mathbb{R} \longrightarrow [0, \infty)$ as:*
$$g(x) = \sum_{r=1}^{n} q\,|x - a_r - t|\,, x \in \mathbb{R}.$$
Then

1. *If n is odd, g attains its smallest value at $\left(t + a_{\frac{n+1}{2}}\right)$.*
2. *If n is even, g attains its smallest value at any $x \in \left[t + a_{\frac{n}{2}}, t + a_{\frac{n+2}{2}}\right]$.*

Proof. Since $\{a_r\}_{r=1}^{n}$ is strictly increasing, $\{t + a_r\}_{r=1}^{n}$is so. Since $g(x) = \sum\limits_{r=1}^{n} q\,|x - (a_r + t)|$ for each $x \in \mathbb{R}$, the result follows by Corollary 2.8. □

Next, we give a generalization of Theorem 2.1.

Corollary 2.9. *Let s, b be positive numbers, and let $A = \{a_r\}_{r=1}^{n}$ be a strictly increasing sequence of real numbers, where $n > 1$ is an integer. Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a strictly increasing continuous function, and suppose that L_f is the function defined on $\mathbb{R}$ as follows:*
$$L_f(x) = \sum_{r=1}^{n} T_{s,b}\left(f(x) - f(a_r)\right).$$
Then

1. *If $[kb - (n-k)s] = 0$ for some $1 \le k < n$, L_f attains its smallest value at any value $t \in [a_k, a_{k+1}]$.*
2. *If $b - (n-1)s > 0$, L_f attains its smallest value at a_1.*
3. *If $(n-1)b - s < 0$, L_f attains its smallest value at a_n.*
4. *If $[(k-1)b - (n-k+1)s][kb - (n-k)s] < 0$ for some $1 < k < n$, L_f attains its smallest value at a_k*

Proof. Since f is strictly increasing, we conclude that $B = \{f(a_r)\}_{r=1}^{n}$ is strictly increasing. Clearly, $L_f(x) = L_{s,b,B}(f(x))$ for each $x \in \mathbb{R}$. Now, we consider the four cases:

1. If $[kb - (n-k)s] = 0$ for some $1 \le k < n$, $L_{s,b,B}(x)$ attains its smallest value at any value $t \in [f(a_k), f(a_{k+1})]$. Since f has a continuous strictly increasing inverse, we conclude that $L_{s,b,B}(f(x))$ attains its smallest value at any value $t \in [a_k, a_{k+1}]$.
2. If $b - (n-1)s > 0$, $L_{s,b,B}(x)$ attains its smallest value at $f(a_1)$, which implies that $L_{s,b,B}(f(x))$ attains its smallest value ata_1.
3. If $(n-1)b - s < 0$, $L_{s,b,B}(x)$ attains its smallest value at $f(a_n)$, which implies that $L_{s,b,B}(f(x))$ attains its smallest value ata_n.
4. If $[(k-1)b - (n-k+1)s][kb - (n-k)s] < 0$ for some $1 < k < n$, $L_{s,b,B}(x)$ attains its smallest value at $f(a_k)$, which implies that $L_{s,b,B}(f(x))$ attains its smallest value ata_k.

□

Now, we introduce the second main result in the present paper.

Theorem 2.10. *Let $A = \{a_r\}_{r=1}^{n}$ be a strictly increasing sequence of real numbers, where $n > 1$ is an integer, and let $C = \{c_r\}_{r=1}^{n}$ be a sequence of positive real numbers. Define $W_{A,C} : \mathbb{R} \longrightarrow [0, \infty)$ as:*

$$W_{A,C}(x) = \sum_{r=1}^{n} c_r |x - a_r|, \; x \in \mathbb{R}.$$

Then

1. *If $\sum_{r=k+1}^{n} c_r - \sum_{r=1}^{k} c_r = 0$ for some $1 \leq k < n$, $W_{A,C}$ attains its smallest value at any value $t \in [a_k, a_{k+1}]$.*
2. *If $\sum_{r=2}^{n} c_r - c_1 < 0$, $W_{A,C}$ attains its smallest value at a_1.*
3. *If $c_n - \sum_{r=1}^{n-1} c_r > 0$, $W_{A,C}$ attains its smallest value at a_n.*
4. *If $0 < \sum_{r=1}^{k} c_r - \sum_{r=k+1}^{n} c_r < 2c_k$ for some $1 < k < n$, $W_{A,C}$ attains its smallest value at a_k.*

Proof. Note that for $x \leq a_1$,

$$W_{A,C}(x) = \sum_{r=1}^{n} c_r (a_r - x) = M_1 - x \sum_{r=1}^{n} c_r ,$$

and for $x \geq a_n$,

$$W_{A,C}(x) = \sum_{r=1}^{n} c_r (x - a_r) = x \sum_{r=1}^{n} c_r - M_1 ,$$

where

$$M_1 = \sum_{r=1}^{n} c_r a_r.$$

Also, for $a_{k-1} \leq x \leq a_k$,where $1 < k < n$, we have

$$W_{A,C}(x) = M_k - x \left(\sum_{r=k}^{n} c_r - \sum_{r=1}^{k-1} c_r \right), \text{ where } M_k = \sum_{r=k}^{n} c_r a_r - \sum_{r=1}^{k-1} c_r a_r.$$

Clearly, $W_{A,C}$ is decreasing over $(-\infty, a_1]$ and is increasing over $[a_n, \infty)$. In addition, the sequence $\left(\sum_{r=k}^{n} c_r - \sum_{r=1}^{k-1} c_r \right)_{k=2}^{n}$ is strictly decreasing. Now, we consider the following cases:

1. $\sum_{r=k+1}^{n} c_r - \sum_{r=1}^{k} c_r = 0$ for some $1 \leq k < n$.

   If $k = 1$ then $W_{A,C}$ is constant over $[a_1, a_2]$. Since $\left(\sum_{r=l}^{n} c_r - \sum_{r=1}^{l-1} c_r \right)_{l=2}^{n}$ is strictly decreasing, $W_{A,C}$ is increasing over $[a_2, a_n]$, and since $W_{A,C}$ is increasing over $[a_n, \infty)$,we conclude that $W_{A,C}$ is increasing over $[a_2, \infty)$. Since $W_{A,C}$ is decreasing over $(-\infty, a_1]$, We conclude that $W_{A,C}$ attains its smallest value at any $t \in [a_1, a_2]$.

   If $k = n-1$ then $W_{A,C}$ is constant over $[a_{n-1}, a_n]$. Since$\left(\sum_{r=l}^{n} c_r - \sum_{r=1}^{l-1} c_r \right)_{l=2}^{n}$ is strictly decreasing, $W_{A,C}$ is decreasing over $[a_1, a_{n-1}]$. Since $W_{A,C}$ is decreasing over $(-\infty, a_1]$ and is increasing over $[a_n, \infty)$, $W_{A,C}$ attains its smallest value at any $t \in [a_{n-1}, a_n]$.

   If $1 < k < n-1$ then $W_{A,C}$ is constant over $[a_k, a_{k+1}]$. Since$\left(\sum_{r=l}^{n} c_r - \sum_{r=1}^{l-1} c_r \right)_{l=2}^{n}$ is strictly decreasing, $W_{A,C}$ is decreasing over $[a_1, a_k]$ and is increasing over $[a_{k+1}, a_n]$. Since $W_{A,C}$ is decreasing over $(-\infty, a_1]$ and is increasing over $[a_n, \infty)$, $W_{A,C}$ attains its smallest value at any $t \in [a_k, a_{k+1}]$.

2. $\sum_{r=2}^{n} c_r - c_1 < 0$.
   Since$\left(\sum_{r=l}^{n} c_r - \sum_{r=1}^{l-1} c_r\right)_{l=2}^{n}$ is strictly decreasing, $W_{A,C}$ is increasing over $[a_1, a_n]$. Since $W_{A,C}$ is decreasing over $(-\infty, a_1]$ and is increasing over $[a_n, \infty)$, $W_{A,C}$ attains its smallest value at a_1.
3. $c_n - \sum_{r=1}^{n-1} c_r > 0$.
   Since$\left(\sum_{r=l}^{n} c_r - \sum_{r=1}^{l-1} c_r\right)_{l=2}^{n}$ is strictly decreasing, $W_{A,C}$ is decreasing over $[a_1, a_n]$. Since $W_{A,C}$ is decreasing over $(-\infty, a_1]$ and is increasing over $[a_n, \infty)$, $W_{A,C}$ attains its smallest value at a_n.
4. $0 < \sum_{r=1}^{k} c_r - \sum_{r=k+1}^{n} c_r < 2c_k$ for some $1 < k < n$.
   Note that
$$\sum_{r=k}^{n} c_r - \sum_{r=1}^{k-1} c_r = 2c_k + \sum_{r=k+1}^{n} c_r - \sum_{r=1}^{k} c_r.$$
   Since $0 < \sum_{r=1}^{k} c_r - \sum_{r=k+1}^{n} c_r < 2c_k$,we conclude that
$$\sum_{r=k}^{n} c_r - \sum_{r=1}^{k-1} c_r > 0 \text{ and } \sum_{r=k+1}^{n} c_r - \sum_{r=1}^{k} c_r < 0.$$
   Since$\left(\sum_{r=l}^{n} c_r - \sum_{r=1}^{l-1} c_r\right)_{l=2}^{n}$ is strictly decreasing,we conclude that $W_{A,C}$ is decreasing over $[a_1, a_k]$ and is increasing over $[a_k, a_n]$. Since $W_{A,C}$ is decreasing over $(-\infty, a_1]$ and is increasing over $[a_n, \infty)$, $W_{A,C}$ attains its smallest value at a_k.

□

Now, let us give a numerical example that supports Theorem 2.10.

Example 2.11. *Suppose that we want to minimize the function* $g(x) = 2|x+0.5| + |x-0.5| + 3|x-1| + |x-2| + 2|x-2.5| + |x-3|$. *Clearly,* $g = W_{A,C}$, *where* $A = \{a_r\}_{r=1}^{6}$ *with* $a_1 = -0.5 < a_2 = 0.5 < a_3 = 1 < a_4 = 2 < a_5 = 2.5 < a_6 = 3$, *and* $C = \{c_r\}_{r=1}^{6}$*with* $c_1 = 2, c_2 = 1, c_3 = 3, c_4 = 1, c_5 = 2, c_6 = 1$. *Note that*
$$0 < \sum_{r=1}^{3} c_r - \sum_{r=4}^{6} c_r = 6 - 4 = 1 < 2c_3 = 6.$$
By Theorem 2.10 (Part 4), we conclude that g *attains its smallest value at* $a_3 = 1$. *Graphing the function* g *, we get the same conclusion (See Figure 1 below).*

The next Corollary shows a more general form of Theorem 2.10.

Corollary 2.12. *Let* $A = \{a_r\}_{r=1}^{n}$ *be a strictly increasing sequence of real numbers, where* $n > 1$ *is an integer, and let* $C = \{c_r\}_{r=1}^{n}$ *be a sequence of positive real numbers. Suppose that* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *is a strictly increasing continuous function, and suppose that* W_f *is the function defined on* $\mathbb{R}$ *as follows:*
$$W_f(x) = \sum_{r=1}^{n} c_r |f(x) - f(a_r)|.$$
Then

1. *If* $\sum_{r=k+1}^{n} c_r - \sum_{r=1}^{k} c_r = 0$ *for some* $1 \le k < n$, W_f *attains its smallest value at any value* $t \in [a_k, a_{k+1}]$.
2. *If* $\sum_{r=2}^{n} c_r - c_1 < 0$, W_f *attains its smallest value at* a_1.

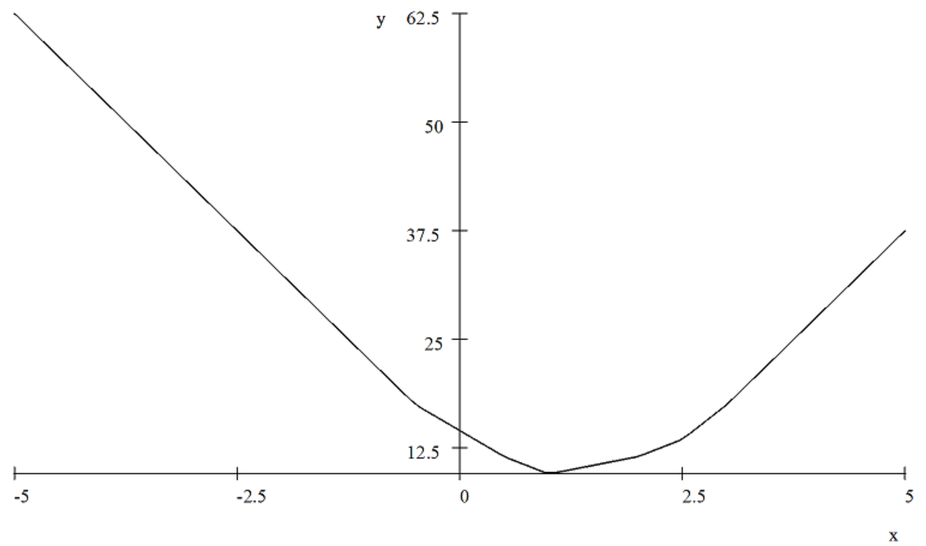

Figure 1: Graph of $g(x)$

3. If $c_n - \sum_{r=1}^{n-1} c_r > 0$, W_f *attains its smallest value at* a_n.

4. If $0 < \sum_{r=1}^{k} c_r - \sum_{r=k+1}^{n} c_r < 2c_k$ *for some* $1 < k < n$, W_f *attains its smallest value at* a_k.

Proof. Since f is strictly increasing, we conclude that $B = \{f(a_r)\}_{r=1}^{n}$ is strictly increasing. Clearly, $W_f(x) = W_{B,C}(f(x))$ for each $x \in \mathbb{R}$, as in Theorem 2.10 Now, we consider the four cases:

1. If $\sum_{r=k+1}^{n} c_r - \sum_{r=1}^{k} c_r = 0$ for some $1 \leq k < n$, $W_{B,C}(x)$ attains its smallest value at any value $t \in [f(a_k), f(a_{k+1})]$. Since f has strictly increasing continuous inverse, we conclude that $W_{B,C}(f(x))$ attains its smallest value at any value $t \in [a_k, a_{k+1}]$.
2. If $\sum_{r=2}^{n} c_r - c_1 < 0$, $W_{B,C}(x)$ attains its smallest value at $f(a_1)$, which implies that $W_{B,C}(f(x))$ attains its smallest value at a_1.
3. If $c_n - \sum_{r=1}^{n-1} c_r > 0$, $W_{B,C}(x)$ attains its smallest value at $f(a_n)$, which implies that $W_{B,C}(f(x))$ attains its smallest value at a_n.
4. If $0 < \sum_{r=1}^{k} c_r - \sum_{r=k+1}^{n} c_r < 2c_k$ for some $1 < k < n$, $W_{B,C}(x)$ attains its smallest value at $f(a_k)$, which implies that $W_{B,C}(f(x))$ attains its smallest value at a_k.

□

Now, we introduce the third main result in the present paper.

Theorem 2.13. *Let* $t > 0$ *and* $\{a_r\}_{r=1}^{n}$ *be a strictly increasing sequence of real numbers, where* $n > 1$ *is an integer. Suppose that* $2t < a_r - a_{r-1}$ *for each* $1 < r \leq n$. *Define* $G, \Psi_t : \mathbb{R} \longrightarrow [0, \infty)$ *as:*

$$\Psi_t(x) = \begin{cases} |x-t| & \text{If } x \geqslant 0 \\ |x+t| & \text{If } x < 0 \end{cases},$$

and

$$G(x) = \sum_{r=1}^{n} \Psi_t(x - a_r), x \in \mathbb{R}.$$

Then

1. *If n is odd, G attains its smallest value at any* $x \in \left\{ a_{\frac{n+1}{2}} + t, a_{\frac{n+1}{2}} - t \right\}$.
2. *If n is even, G attains its smallest value at any* $x \in \left[a_{\frac{n}{2}} + t, a_{\frac{n+2}{2}} - t \right]$.

Proof. Since $2t < a_r - a_{r-1}$ for each $1 < r \leq n$, we conclude that for $x \leq a_1 - t$,

$$G(x) = \sum_{r=1}^{n} |x - (a_r - t)| = M_{0,1} - nx, \text{ where } M_{0,1} \text{ is constant,}$$

and for $x \geq a_n + t$,

$$G(x) = \sum_{r=1}^{n} |x - (a_r + t)| = M_{0,n} + nx, \text{ where} M_{0,n} \text{ is constant.}$$

Also, for $1 \leq k < n$,

$$\begin{aligned} G(x) &= \sum_{r=1}^{k} |x - (a_r + t)| + \sum_{r=k+1}^{n} |x - (a_r - t)| \\ &= M_{k,1} - (n - 2(k-1))\, x \quad \text{for } x \in [a_k, a_k + t], \end{aligned}$$

$$\begin{aligned} G(x) &= \sum_{r=1}^{k} |x - (a_r + t)| + \sum_{r=k+1}^{n} |x - (a_r - t)| \\ &= M_{k,2} - (n - 2k)\, x \quad \text{for } x \in [a_k + t, a_{k+1} - t], \end{aligned}$$

and

$$\begin{aligned} G(x) &= \sum_{r=1}^{k} |x - (a_r + t)| + \sum_{r=k+1}^{n} |x - (a_r - t)| \\ &= M_{k,3} - (n - 2(k+1))\, x \quad \text{for } x \in [a_{k+1} - t, a_{k+1}], \end{aligned}$$

where $M_{k,1}, M_{k,2}$ and $M_{k,3}$ are constants.

Clearly, G is decreasing over $(-\infty, a_1 - t]$, and is increasing over$[a_n + t, \infty)$. Now, we consider two cases:

1. If n is odd, $n = 2L + 1$ for some positive integer L. Note that G is decreasing over $[a_k, a_{k+1} - t]$ for each $1 \leq k \leq L$, and over $[a_1 - t, a_1]$, which implies that G is decreasing over $(-\infty, a_{L+1} - t]$. On the other hand, G is increasing over $[a_k + t, a_{k+1}]$ for each $L+1 \leq k < n$, and over $[a_n, a_n + t]$, which implies that G is increasing over $[a_{L+1} + t, \infty)$. Since G is increasing over $[a_{L+1} - t, a_{L+1}]$ and is decreasing over $[a_{L+1}, a_{L+1} + t]$, we conclude that G attains its smallest value at $a_{L+1} - t$ or at $a_{L+1} + t$. But

$$\begin{aligned} G(a_{L+1} - t) &= \sum_{r=1}^{L} |a_{L+1} - t - (a_r + t)| + \sum_{r=L+1}^{n} |a_{L+1} - t - (a_r - t)| \\ &= \sum_{r=1}^{L} (a_{L+1} - a_r - 2t) + \sum_{r=L+2}^{n} (a_r - a_{L+1}) \\ &= (L - (n - L - 1))\, a_{L+1} - 2tL + \sum_{r=L+2}^{n} a_r - \sum_{r=1}^{L} a_r \\ &= -2tL + \sum_{r=L+2}^{n} a_r - \sum_{r=1}^{L} a_r, \end{aligned}$$

and

$$\begin{aligned} G\left(a_{L+1}+t\right) &= \sum_{r=1}^{L+1}\left|a_{L+1}+t-\left(a_r+t\right)\right|+\sum_{r=L+2}^{n}\left|a_{L+1}+t-\left(a_r-t\right)\right| \\ &= \sum_{r=1}^{L}\left(a_{L+1}-a_r\right)+\sum_{r=L+2}^{n}\left(a_r-a_{L+1}-2t\right) \\ &= \left(L-\left(n-L-1\right)\right)a_{L+1}-2t\left(n-L-1\right)+\sum_{r=L+2}^{n}a_r-\sum_{r=1}^{L}a_r \\ &= -2tL+\sum_{r=L+2}^{n}a_r-\sum_{r=1}^{L}a_r, \end{aligned}$$

which implies that $G\left(a_{L+1}-t\right)=G\left(a_{L+1}+t\right)$. Therefore, G attains its smallest value at any $x\in\left\{a_{\frac{n+1}{2}}+t, a_{\frac{n+1}{2}}-t\right\}$.

2. If n is even, $n=2L$ for some positive integer L. Note that

$$\begin{aligned} G\left(x\right) &= M_{L,2} \text{ on } \left[a_L+t, a_{L+1}-t\right], \\ G\left(x\right) &= M_{L-1,3} \text{ on } \left[a_L-t, a_L\right], \end{aligned}$$

and

$$G\left(x\right)=M_{L+1,1} \; on \left[a_{L+1}, a_{L+1}+t\right].$$

Also, G is decreasing over $[a_{k-1}, a_k-t]$ for each $1<k\leq L$, and over $[a_1-t, a_1]$, which implies that G is decreasing over $\left(-\infty, a_L-t\right]$. On the other hand, G is increasing over $[a_k+t, a_{k+1}]$ for each $L+1\leq k<n$, and over $[a_n, a_n+t]$, which implies that G is increasing over $\left[a_{L+1}+t, \infty\right)$. Since G is decreasing over $[a_L, a_L+t]$ and is increasing over $\left[a_{L+1}-t, a_{L+1}\right]$, we conclude that G attains its smallest value at any $x\in[a_L+t, a_{L+1}-t]=\left[a_{\frac{n}{2}}+t, a_{\frac{n+2}{2}}-t\right]$.

□

Now, let us give a numerical example that supports Theorem 2.13.

Example 2.14. *Suppose that we want to minimize the function* $G\left(x\right)=\sum_{r=1}^{5}\Psi_t\left(x-a_r\right)$, *where* $a_1=-3<a_2=-1<a_3=1.5<a_4=3<a_5=5$, *and* $t=0.5$. *Note that* $2t<a_r-a_{r-1}$ *for each* $1<r\leq n$. *Since* $n=5$ *is odd, Theorem 2.13 implies that* G *attains its smallest value at* $x=2$ *and* $x=1$. *Graphing the function* G *, we get the same conclusion (See Figure 2 below).*

3 Some Applications

In this section, we discuss some applications of the results obtained.

Example 3.1. *Let* $N\geq 2$ *be a positive integer. For each positive integer* k, *let* $C^{(k)}=\left\{c_i^{(k)}\right\}_{i=1}^{N}$ *be the sequence of real numbers which is defined as:*

$$c_i^{(k)}=\begin{cases} i & if\; i<k \\ \left|\frac{(N-k)(N-k+1)}{2}-\frac{k(k-1)}{2}\right|+1 & if\; i=k \\ n-i+1 & if\; i>k \end{cases}, \; for\; 1\leq i\leq N.$$

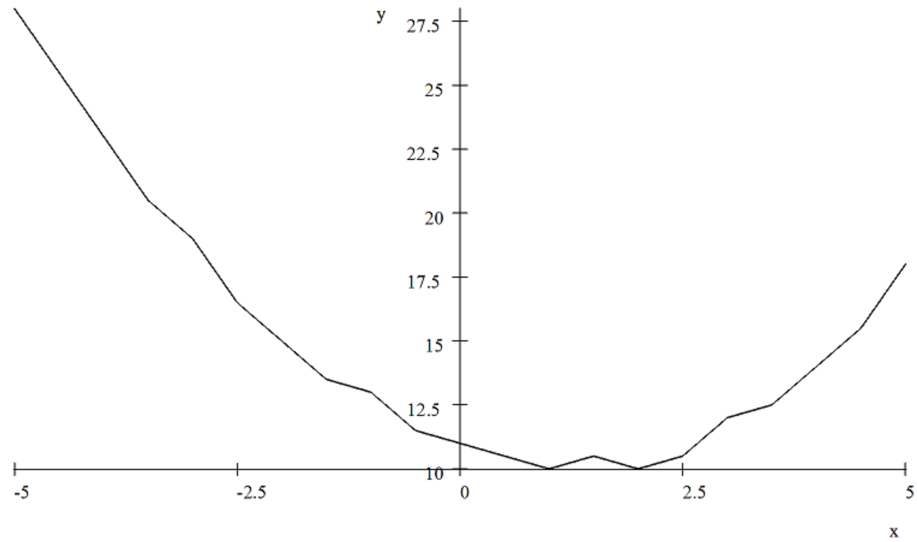

Figure 2: Graph of $G(x)$

For each positive integer k *and each strictly increasing sequence of real numbers* $A = \{a_i\}_{i=1}^{N}$ *,define* $S_{A,k} : \mathbb{R} \longrightarrow [0, \infty)$ *as follows:*

$$S_{A,k}(x) = \sum_{i=1}^{N} c_i^{(k)} |x - a_i| \text{ for } x \in \mathbb{R}.$$

Note that for each positive integer k *,we have*

$$\begin{aligned}\sum_{i=1}^{k} c_i^{(k)} - \sum_{i=k+1}^{N} c_i^{(k)} &= c_k^{(k)} + \sum_{i=1}^{k-1} i - \sum_{i=k+1}^{N} (n - i + 1) \\ &= c_k^{(k)} + \frac{k(k-1)}{2} - (N+1)(N-k) + \frac{N(N+1)}{2} - \frac{k(k+1)}{2} \\ &< 2c_k^{(k)}.\end{aligned}$$

Also, it is not hard to see that $\sum_{i=1}^{k} c_i^{(k)} - \sum_{i=k+1}^{N} c_i^{(k)} > 0$*. By Theorem 2.10, we conclude that for each positive integer* k *and each strictly increasing sequence of real numbers* $A = \{a_i\}_{i=1}^{N}$ *,* $S_{A,k}$ *attains its smallest value at* a_k*.*

The next example shows and application of Corollary 2.12 in the context of weighted l_1-norm of real vector-valued functions defined on $\mathbb{R}$.

Example 3.2. *Consider the curve* $r : \mathbb{R} \longrightarrow \mathbb{R}^4$ *defined as*

$$r(x) = (2^x - 8, 2^x - 4, 2^x - 16, 2^x - 2).$$

The weighted l_1*-norm of* $r(x)$ *for the weights* $C = (w_1, w_2, w_3, w_4)$ *is defined as*

$$\begin{aligned}\|r(x)\|_{l_1^w} &= w_1 |2^x - 8| + w_2 |2^x - 4| + w_3 |2^x - 16| + w_4 |2^x - 2| \\ &= w_4 |2^x - 2| + w_2 \left|2^x - 2^2\right| + w_1 \left|2^x - 2^3\right| + w_3 \left|2^x - 2^4\right|.\end{aligned}$$

In particular, if we consider the weights $C = (2, 3, 6, 2)$ *, then*

$$0 < w_4 + w_2 + w_1 - w_3 = 2 + 3 + 2 - 6 = 1 < 2w_1.$$

Using Corollary 2.12, we conclude that $\|r(x)\|_{l_1^w}$ *attains its smallest value at* $x = 3$*, and it is equal to* 72.

Example 3.3. *Let* s, b *be positive numbers such that* $s = mb$*, where* m *is a positive integer. If* $n = m + 1$ *then*

$$mb - (n - m)s = mb - s = 0.$$

By Theorem 2.1, for each strictly increasing sequence of real numbers $A = \{a_r\}_{r=1}^n$ *,* $L_{s,b,A}$ *attains its smallest value at any* $t \in [a_m, a_{m+1}]$ *. Also, By Corollary 2.7,* $L_{b,s,A}$ *attains its smallest value at any* $t \in [a_1, a_2]$ *.*

We close the section by the following Corollary which is an application of Corollary 2.12 in the context of integrals.

Corollary 3.4. *Let* $h : \mathbb{R} \longrightarrow \mathbb{R}$ *be a non-negative continuous function. Let* $A = \{a_r\}_{r=1}^n$ *be a strictly increasing sequence of real numbers, where* $n > 1$ *is an integer, and let* $\{b_r\}_{r=1}^n$ *be a sequence of non-zero real numbers.. Define* $g : \mathbb{R} \longrightarrow [0, \infty)$ *as:*

$$g(x) = \sum_{r=1}^{n} \left| b_i \int_{a_r}^{x} h(t)\, dt \right|, x \in \mathbb{R}.$$

Then

1. *If* $\sum_{r=k+1}^{n} |b_r| - \sum_{r=1}^{k} |b_r| = 0$ *for some* $1 \le k < n$*,* g *attains its smallest value at any value* $t \in [a_k, a_{k+1}]$ *.*
2. *If* $\sum_{r=2}^{n} |b_r| - |b_1| < 0$*,* g *attains its smallest value at* a_1*.*
3. *If* $|b_n| - \sum_{r=1}^{n-1} |b_r| > 0$*,* g *attains its smallest value at* a_n*.*
4. *If* $0 < \sum_{r=1}^{k} |b_r| - \sum_{r=k+1}^{n} |b_r| < 2|b_k|$ *for some* $1 < k < n$*,* g *attains its smallest value at* a_k*.*

Proof. Note that

$$g(x) = \sum_{r=1}^{n} |b_r| \left| \int_0^x h(t)\, dt - \int_0^{a_r} h(t)\, dt \right|.$$

Since $\int_0^x h(t)\, dt$ is a strictly increasing continuous function over $\mathbb{R}$, the result follows by using Corollary 2.12. □

4 Conclusion

In conclusion, the importance of studying Problem 1 and Problem 2 in the present paper is due to their applications in different fields of study such as Quantile Analysis and Approximation T heory. In addition, they are equivalent to constrained separable problems over the whole space. The results obtained in the present paper can be used to solve other problems in different contexts such as the ones introduced in the previous section. Also, they can be generalized to include more optimization problems in different aspects. Some new generalizations will be sought in future projects.

Acknowledgment

The authors extend their appreciation to Deanship of scientific research at king Saud University, for funding the work.

Competing Interests

The authors declare that no competing interests exist.

References

[1] El-Attar RA, Vidyasagar M, Dutta SRK. An algorithm for l_1-norm minimization with application to nonlinear l_1-approximation. SIAM Journal on Numerical Analysis. 2004;16:70-86.

[2] Han-Lin L, Chian-Son Y. A global optimization method for nonconvex separable programming problems. European Journal of Operational Research. 1999;117:275-292.

[3] Khashshan MM. Minimizing the sum of linear absolute value functions on $\mathbb{R}$. International Journal of Computational and Applied Mathematics. 2010;5:537-540.

[4] Khashshan MM, Obeidat ST. Minimizing a special type of functions. Applied Mathematical Sciences. 2014;8:1099-1107.

[5] Huang C. An effective linear approximation method for separable programming problems. Applied Mathematics and Computation. 2009;215:1496-1506.

[6] Zhang H, Wang S. Global optimization of separable objective functions on convex polyhedra via piecewise-linear approximation. Journal of Computational and Applied Mathematics. 2006;197:212-217.

[7] Melachrinoudis E. An analytical solution to the minimum Lp -Norm of a Hyperplane. Journal of Mathematical Analysis and Applications. 1997;211:172-189.

[8] Melachrinoudis E, Xanthopulos Z. A maximum Lp distance Problem. Journal of Mathematical Analysis and Applications. 1998;217:650-671.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
www.sciencedomain.org/review-history.php?iid=1143&id=6&aid=9338