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A Phenomenon in Fibonacci Numbers and Its Generalization

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Abstract

Motivated by the result of Fibonacci numbers for which the ratio of successive terms tends to a limit, which is commonly known as the Golden Ratio, we prove an immediate generalization for a wider class of recurrence sequences. We note that such limiting behavior for ratio of successive terms of general linear recurrence sequences has been well discussed, but still they need to satisfy specific conditions for the limit to exist. Our contribution is that we show that such conditions are indeed satisfied for the cases we are considering. For an application of our main result, we find a natural way to approximate an algebraic number, which is a zero for some class of polynomial equations, by rational numbers. As recently there seem to be renewed interests on Fibonacci numbers and related recurrence sequences, we hope that our elementary methods and results may shed some light for solving the related problems.

Keywords: Fibonacci numbers; recurrence sequences; generating functions.

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1 Introduction

We recall that the Fibonacci numbers are defined inductively by $F_1 = 1, F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. This interesting sequence arose prominently from early on in the history in many different areas including mathematics, natural sciences, visual arts, architecture, music, and so on ([1], [2], [3], [4], [5]). Let $\alpha_n = F_{n+1}/F_n$. In this section we will give four different proofs for the following proposition showing that $\lim_{n\to\infty} \alpha_n$ exists, which serves as our initial motivation for generalizing this phenomenon to other (possibly higher) recurrence sequences. Our main theorem is given in the second section. In section 3, we conclude by giving some remarks, applications, and pointing out some related (seemingly) open problems. For basic properties about Fibonacci and other recurrence sequences, and for the basic theory of continued fractions, we refer the readers to [6]. We mention in passing that the study of recurrence sequences and continued fractions has long attracted the attention of leading mathematicians, computer scientists, and other researchers, hence there has been an extensive literature (to name just a few, [7], [8], [9], [2], and a recent talk entitled "Alf van der Poorten and Continued Fractions" by J. Shallit); amazingly this continues to grow, as can be seen from some recent literature (see for example, [10], [11], [12], [13], [14]) and many recent online preprints available by keyword search. We made our presentation self-contained by providing direct arguments for some existing results such as the Binet type formulas.

Proposition 1.1. *The limit of α_n as $n \to \infty$ exists, and in fact*

$$\lim_{n\to\infty} \alpha_n = \frac{1+\sqrt{5}}{2}.$$

First proof. Clearly $\alpha_{n+1} = \frac{F_{n+2}}{F_{n+1}} = \frac{F_{n+1}+F_n}{F_{n+1}} = 1 + \frac{1}{\frac{F_{n+1}}{F_n}} = 1 + \frac{1}{\alpha_n}$. Since $\alpha_1 = 1$, it follows that α_n's are simply the convergents of the continued fraction $[1; 1, 1, 1, \cdots]$. For example, $\alpha_2 = 1 + \frac{1}{1}, \alpha_3 = 1 + \frac{1}{1+\frac{1}{1}}$. By the general theory of continued fractions ([6]), the limit of α_n as $n \to \infty$ exists. The limit can be easily solved from the equation $x = 1 + \frac{1}{x}$. A positive solution gives $x = \frac{1+\sqrt{5}}{2}$.

Second proof. The recursion $F_{n+2} = F_{n+1} + F_n$ with initial conditions $F_1 = F_2 = 1$ can be solved directly: Substituting $F_n = z^n$ into the relation yields $z^2 = z + 1$. The equation $z^2 - z - 1 = 0$ has roots $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$. Then by linearity and the initial conditions, the solution of the recursion is given by

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Then it is straightforward to show that $\lim F_{n+1}/F_n$ exists and equals $\frac{1+\sqrt{5}}{2}$.

Third proof. As in the first proof, $\alpha_{n+1} = 1 + \frac{1}{\alpha_n}$ for $n \geq 1$. Then we also have $\alpha_{n+2} = 1 + \frac{1}{\alpha_{n+1}}$. Subtracting gives

$$\alpha_{n+2} - \alpha_{n+1} = \frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}\alpha_n}.$$

It follows that

$$|\alpha_{n+2} - \alpha_{n+1}| = \frac{1}{\alpha_{n+1}\alpha_n}|\alpha_{n+1} - \alpha_n|.$$

Now by induction, it is easy to establish that

$$\frac{3}{2} \leq \alpha_n \leq 2 \text{ for } n \geq 3.$$

This shows that the contraction factor $\frac{1}{\alpha_{n+1}\alpha_n}$ is less than or equal to $\left(\frac{2}{3}\right)^2 < 1$. Then by standard argument, α_n is a Cauchy sequence, and therefore $\lim_{n\to\infty}\alpha_n$ exists. Solving $x = 1 + \frac{1}{x}$ yields a positive solution $x = \frac{1+\sqrt{5}}{2}$. Alternatively, we note that the mapping $f : [\frac{3}{2}, 2] \to [\frac{3}{2}, 2]$ defined by $f(x) = 1+\frac{1}{x}$ is a contraction mapping (see for example [15]), so the unique fixed point can be found by taking limit of $f^n(x_0)$ for any initial point $x_0 \in [\frac{3}{2}, 2]$ as $n \to \infty$.

Fourth proof. Let $f(x) = a_0+a_1x+a_2x^2+\cdots$ be the generating function for the Fibonacci numbers (here $a_0 = 0, a_i = F_i$ for $i \geq 1$ as defined above). We can easily find $f(x)$ by using the recursive relation

$$a_{n+2} = a_{n+1} + a_n,\ n \geq 0.$$

Namely by summing n from 0 to ∞ of the following relation

$$\sum_{n\geq 0} a_{n+2}x^{n+2} = \sum_{n\geq 0} a_{n+1}x^{n+2} + \sum_{n\geq 0} a_n x^{n+2},$$

we see that

$$f(x) - a_0 - a_1x = x(f(x) - a_0) + x^2 f(x),$$

which yields

$$f(x) = \frac{a_0 + (a_1 - a_0)x}{1 - x - x^2} = \frac{x}{1 - x - x^2} = \frac{x}{(1 - \lambda_1 x)(1 - \lambda_2 x)}, \tag{1}$$

where $\lambda_1 = \frac{1+\sqrt{5}}{2}, \lambda_2 = \frac{1-\sqrt{5}}{2}$ are the reciprocal roots of the polynomial equation $1 - x - x^2 = 0$. By partial fraction expansion, we have that

$$\frac{x}{(1 - \lambda_1 x)(1 - \lambda_2 x)} = \frac{\frac{1}{\lambda_1-\lambda_2}}{1 - \lambda_1 x} + \frac{\frac{1}{\lambda_2-\lambda_1}}{1 - \lambda_2 x}.$$

Expanding this in power series in (1) and equating coefficient of x^n, we see immediately that

$$a_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}.$$

This essentially is the solution from the second proof, so we know

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda_1 = \frac{1 + \sqrt{5}}{2}.$$

Observation 1.1. By the ratio test, the radius of convergence for the power series of $f(x)$ is given by $\lim\left|\frac{a_n}{a_{n+1}}\right|$, provided that the limit exists. This is related to the poles of $f(x)$ of minimum absolute value, which in the above case is $\frac{1}{\lambda_1}$. So if $\lim\frac{a_{n+1}}{a_n}$ exists, then the answer would be λ_1. The more difficult part of the question is why the limit exists.

Our goal in the next section is to generalize this phenomenon. We first look at the following examples.

Example 1.1. *Let $a_{n+2} = \alpha a_{n+1} + \beta a_n$, where $\alpha, \beta > 0$ and $a_0 = 1, a_1 = \alpha$. Then it is easy to see that $f(x) = \frac{1}{1-\alpha x-\beta x^2}$. We claim that $\lim\frac{a_{n+1}}{a_n} = \frac{1}{x_0}$, where x_0 is the unique positive root of $1 - \alpha x - \beta x^2 = 0$.*

Proof. Let y_0 (necessarily negative) be the other root of $1 - \alpha x - \beta x^2 = 0$. If we let $\lambda_1 = \frac{1}{x_0}$, and $\lambda_2 = \frac{1}{y_0}$, then by the fourth proof, the solution a_n is of the form $a_n = k_1\lambda_1^n + k_2\lambda_2^n$, where k_1, k_2 are constants depending on the initial conditions (one can argue that $k_1 \neq 0$ because $f(x)$ has a pole at $x = x_0$). Therefore it suffices to show that $|y_0| > x_0$. For this, rewrite the equations as

$$\begin{matrix} y_0^2 + \frac{\alpha}{\beta}y_0 - \frac{1}{\beta} = 0 \\ x_0^2 + \frac{\alpha}{\beta}x_0 - \frac{1}{\beta} = 0 \end{matrix}\ .$$

Subtracting and simplifying, this gives

$$y_0^2 - x_0^2 = -\frac{\alpha}{\beta}(y_0 - x_0) > 0 \Rightarrow |y_0| > x_0.$$

$\square$

Example 1.2. *Let $a_{n+2} = \alpha a_{n+1} + \beta a_n$ with $\alpha = 2, \beta = 3, a_0 = 1$, and $a_1 = 2$. Then by the previous example, we see that $\lim \frac{a_{n+1}}{a_n} = 3$. Note that this can also be seen by using the generalized continued fraction:*

$$\begin{aligned} a_{n+2} &= 2a_{n+1} + 3a_n \\ \Rightarrow \frac{a_{n+2}}{a_{n+1}} &= 2 + \frac{3}{\frac{a_{n+1}}{a_n}}, \end{aligned}$$

which suggests that

$$3 = \lim \frac{a_{n+1}}{a_n} = 2 + \frac{3}{2 + \frac{3}{2+\cdots}}.$$

On the other hand, if we let $\alpha = m > 0, \beta = 1, a_0 = 1$, and $a_1 = m$. Then $\lim \frac{a_{n+1}}{a_n}$ equals the simple continued fraction

$$m + \frac{1}{m + \frac{1}{m+\cdots}}.$$

This was referred to as metallic mean by [16].

2 Generalization

Theorem 2.1. *Let the sequence $\{a_n\}$ be defined by the linear recursion*

$$a_{n+k} = \sum_{i=0}^{k-1} c_i a_{n+i}, c_i > 0, n = 0, 1, 2, \cdots, \tag{2}$$

where $a_0, a_1, \cdots, a_{k-1}$ are nonnegative and not all zero. Then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

exists and equals the unique positive solution of the characteristic polynomial

$$y^k - c_{k-1}y^{k-1} - \cdots - c_1 y - c_0 = 0.$$

For the proof of Theorem 2.1, if we write the generating function of the recurrence sequence by $f(x) = \frac{g(x)}{h(x)}$ (see details below), then the essential point is to show that the unique positive root of the characteristic polynomial has the maximum modulus (i.e. the unique positive root x_0 of $h(x)$ has minimum modulus) and that $g(x)$ does not vanish at $x = x_0$. These facts are established in Lemmas 2.2-2.5. For convenience of the readers we fill in the standard arguments leading to the solutions of linear recurrence relations. Note that the equation $h(x) := 1 - \sum_{i=1}^{k} c_{k-i}x^i = 0$ is obtained from $y^k - \sum_{i=1}^{k} c_{k-i}y^{k-i} = 0$ by applying the transformation $y = \frac{1}{x}$, hence their roots are reciprocal to each other.

Lemma 2.2. *The polynomial equation*

$$1 - \sum_{i=1}^{k} c_{k-i}x^i = 0$$

has a unique positive solution.

Proof. Let $P(x) = 1 - \sum_{i=1}^{k} c_{k-i}x^i$. Then clearly $P(x)$ is a monotone decreasing function for $x \geq 0$ with $P(0) > 0$ and $P(x) < 0$ if x is sufficiently large. Then the result follows from intermediate value theorem for continuous functions. □

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of $\{a_n\}, n \geq 0$. From the definition of recursion (2), it follows that

$$\sum_{n=0}^{\infty} a_{n+k}x^{n+k} = \sum_{n=0}^{\infty}\sum_{i=0}^{k-1} c_i a_{n+i}x^{n+k} = \sum_{i=0}^{k-1} c_i x^{k-i} \sum_{n=0}^{\infty} a_{n+i}x^{n+i},$$

which implies

$$f(x) - a_0 - a_1x - \cdots - a_{k-1}x^{k-1} = c_{k-1}x(f(x) - a_0 - \cdots - a_{k-2}x^{k-2})$$
$$+c_{k-2}x^2(f(x) - a_0 - \cdots - a_{k-3}x^{k-3}) + \cdots + c_1x^{k-1}(f(x) - a_0) + c_0x^k f(x).$$

Rewriting, we see that

$$(1 - c_{k-1}x - \cdots - c_1x^{k-1} - c_0x^k)f(x) = a_0(1 - c_{k-1}x - \cdots - c_1x^{k-1})$$
$$+a_1x(1 - c_{k-1}x - \cdots - c_2x^{k-2}) + \cdots + a_{k-2}x^{k-2}(1 - c_{k-1}x) + a_{k-1}x^{k-1}.$$

We see that

$$f(x) = \frac{g(x)}{1 - c_{k-1}x - \cdots - c_1x^{k-1} - c_0x^k},$$

where

$$g(x) := a_0(1 - c_{k-1}x - \cdots - c_1x^{k-1})$$
$$+a_1x(1 - c_{k-1}x - \cdots - c_2x^{k-2}) + \cdots + a_{k-2}x^{k-2}(1 - c_{k-1}x) + a_{k-1}x^{k-1}.$$

Now define

$$h(x) := 1 - c_{k-1}x - \cdots - c_1x^{k-1} - c_0x^k.$$

Lemma 2.3. *Using the above notations, we have that $g(x_0) \neq 0$, where x_0 is the unique positive solution of $h(x)$, i.e. x_0 is a pole of $f(x)$.*

Proof. Let x_0 be the unique positive zero for the polynomial $h(x)$. Then it is easy to see that $g(x_0) > 0$, by using the relation

$$1 - c_{k-1}x_0 - \cdots - c_1x_0^{k-1} - c_0x_0^k = 0,$$

which implies

$$1 - c_{k-1}x_0 - \cdots - c_1x_0^{k-1} = c_0x_0^k > 0, \text{ etc.}$$

This shows that $f(x) = \frac{g(x)}{h(x)}$ has a pole at $x = x_0$. □

Lemma 2.4. *Let $b_1, \cdots, b_r$ be positive real numbers, ζ a complex number with $|\zeta| = 1$, and let $d_1, \cdots d_r$ be positive integers. Then*

$$|b_1\zeta^{d_1} + \cdots + b_r\zeta^{d_r}| = b_1 + \cdots + b_r \Leftrightarrow \zeta^{\gcd\{d_j - d_i : i<j\}} = 1.$$

Proof. We prove the less obvious direction ($\Rightarrow$). Let $\bar{\zeta}$ be the complex conjugation. We note that $\bar{\zeta} = \frac{1}{\zeta}$.

$$|b_1\zeta^{d_1} + \cdots + b_r\zeta^{d_r}| = b_1 + \cdots + b_r$$
$$\Rightarrow (b_1\zeta^{d_1} + \cdots + b_r\zeta^{d_r})(b_1\bar{\zeta}^{d_1} + \cdots + b_r\bar{\zeta}^{d_r}) = (b_1 + \cdots + b_r)^2$$
$$\Rightarrow b_ib_j(\zeta^{d_i}\bar{\zeta}^{d_j} + \zeta^{d_j}\bar{\zeta}^{d_i}) = 2b_ib_j, \text{ for all } (i,j) \text{ with } i<j$$
$$\Rightarrow \bar{\zeta}^{d_j - d_i} + \zeta^{d_j - d_i} = 2 \text{ for all } (i,j) \text{ with } i<j$$
$$\Rightarrow \zeta^{d_j - d_i} = 1 \text{ for all } (i,j) \text{ with } i<j,$$

from which the result follows. □

Lemma 2.5. *$f(x)$ has no other poles in the closed disk $|z| \leq x_0$.*

Proof. It suffices to show that $h(x)$ has no other zeros in the closed disc $|z| \leq x_0$. Clearly if $|z| \leq x_0$, then

$$\begin{aligned}|h(z)| &= |1 - c_{k-1}z - \cdots - c_1 z^{k-1} - c_0 z^k| \\ &\geq 1 - |c_{k-1}z + \cdots + c_1 z^{k-1} + c_0 z^k| \\ &\geq 1 - c_{k-1}|z| - \cdots - c_1|z|^{k-1} - c_0|z|^k \\ &\geq 1 - c_{k-1}x_0 - \cdots - c_1 x_0^{k-1} - c_0 x_0^k = 0.\end{aligned}$$

In order that $h(z) = 0$, all the above inequalities must be equalities. Necessarily the last (in)equality shows that $|z| = x_0$. If we write $z = x_0\zeta$ with $|\zeta| = 1$, then the equality

$$1 - |c_{k-1}z + \cdots + c_1 z^{k-1} + c_0 z^k| = 1 - c_{k-1}x_0 - \cdots - c_1 x_0^{k-1} - c_0 x_0^k = 0$$

shows $\zeta = 1$ by applying Lemma 2.4 with $b_i = c_{k-i}x_0^i, i = 1, \cdots, k$, and $d_i = i, i = 1, \cdots, k$. Therefore $z = x_0\zeta = x_0$. □

Proof of Theorem 2.1 To prove the theorem, we first consider the case when $h(x)$ has only simple roots. Then

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{g(x)}{h(x)} = \sum_{i=1}^{k} \frac{\alpha_i}{1 - \lambda_i x}, \tag{3}$$

where λ_i's are reciprocal roots of $h(x)$ such that $\lambda_1 := \frac{1}{x_0}$ corresponds to the unique positive root of $y^k - c_{k-1}y^{k-1} - \cdots - c_1 y - c_0 = 0$, and furthermore, $\alpha_1 \neq 0$ because $f(x)$ has a pole at $x = x_0$ by Lemma 2.3. Note that Lemma 2.5 shows that $|\lambda_i| < \lambda_1$ for $i > 1$.

Formula (3) shows that

$$a_n = \sum_{i=1}^{k} \alpha_i \lambda_i^n$$

from which it follows that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda_1 = \frac{1}{x_0}.$$

Now let's deal with the general case. We claim that x_0 is a simple root of $h(x)$. This can be easily checked by noting that $h(x_0)$ and $h'(x_0)$ do not vanish simultaneously, for if they did, then

$$\begin{aligned}1 - c_{k-1}x_0 - c_{k-2}x^2 - \cdots - c_0 x_0^k = 0 \\ -c_{k-1} - 2c_{k-2}x_0 - \cdots - kc_0 x_0^{k-1} = 0.\end{aligned}$$

Multiplying the first equation by $-\frac{1}{x_0}$ and adding to the second equation yields

$$-\frac{1}{x_0} - c_{k-2}x_0 - 2c_{k-3}x_0^2 - \cdots - (k-1)c_0 x_0^{k-1} = 0$$

which is absurd.

Let $\lambda_1, \lambda_2, \cdots, \lambda_r$ be the distinct reciprocal roots of $h(x)$. Then by partial fraction expansion, we can write

$$f(x) = \sum a_n x^n = \frac{\alpha_1}{1 - \lambda_1 x} + \sum_{i=2}^{r} \frac{\beta_{i1}}{1 - \lambda_i x} + \frac{\beta_{i2}}{(1 - \lambda_i x)^2} + \cdots + \frac{\beta_{im_i}}{(1 - \lambda_i x)^{m_i}},$$

where m_i is the multiplicity of λ_i for $2 \leq i \leq r$. Using Taylor's theorem or negative binomial coefficients, we obtain that

$$a_n = \alpha_1 \lambda_1^n + \sum_{i=2}^{r} g_i(n)\lambda_i^n, \tag{4}$$

where $g_i(n)$ is given by

$$\sum_{j=1}^{m_i} \beta_{ij} \binom{n+j-1}{n},$$

where $\binom{n+j-1}{n} = \binom{n+j-1}{j-1} = \frac{(n+j-1)\cdots(n+1)}{(j-1)!}$, if $j \geq 2, = 1$ if $j = 1$. Clearly $g_i(n)$ is a polynomial in n of degree at most $m_i - 1$. Now it is straightforward to see from (4) (noting that $\alpha_1 \neq 0$) that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda_1 = \frac{1}{x_0}.$$

Remark 2.1. The assumption in Theorem 2.1 regarding $c_i > 0$ for all i can be relaxed. For example, it is sufficient to assume that $k = 1, c_0 > 0$, or for $k \geq 2$, $c_i \geq 0$, $c_i > 0$ for at least two indices $0 \leq i \leq k-1$, and $\gcd\{j - i | i < j \text{ with } c_i > 0 \text{ and } c_j > 0\} = 1$. Here $\gcd S$ for a nonempty set $S \neq \{0\}$ of integers means the greatest integer d such that d divides each of the element in S.

Remark 2.2. The restrictions on the initial conditions in Theorem 2.1 for $a_0, \cdots, a_{n-1}$ can be relaxed. By following the proof, it is easy to see that the theorem remains true as long as $g(x_0) \neq 0$, where x_0 is the unique positive root of $h(x)$. Note that the theorem becomes false if $g(x_0) = 0$. For example, consider the linear recurrence $a_{n+3} = a_{n+2} + a_{n+1} + 2a_n, n \geq 0$ with initial conditions $a_0 = 1, a_1 = 0$, and $a_2 = -1$. It can be checked directly that $x_0 = \frac{1}{2}, g(x_0) = h(x_0) = 0$ and $\lim \frac{a_{n+1}}{a_n}$ does not exist. On the other hand, if $a_0 + a_1 + a_2 \neq 0$, then $g(x_0) \neq 0$ and the result in Theorem 2.1 is true.

3 Conclusion

3.1 Applications and Future Work

We illustrate here a way to approximate the unique positive zero of polynomials of the form $h(x) = 1 - c_{k-1}x - \cdots - c_0 x^k$, where for simplicity c_i's are assumed to be integers, so the solution is an algebraic number. The idea is simple: we simply compute the generating function associated with the recursion $a_{n+k} = c_{k-1}a_{n+k-1} + \cdots + c_0 a_n$, and compute the ratios $\frac{a_n}{a_{n+1}}$. These rational numbers converge to the unique positive root of $h(x)$ by Theorem 2.1. For example, from the ratio of coefficients of the generating function associated with the recurrence sequence $a_{n+3} = 2a_{n+2} + 3a_{n+1} + 2a_n$, we find a natural sequence of rational numbers which give the approximation for the unique positive root of the polynomial equation $1 - 2x - 3x^2 - 2x^3 = 0$. We remark here that many questions can be asked. For example, how is the above approximation related to the convergents of the continued fraction associated with the algebraic number? How is the above approximation compared to the Newton's method? Furthermore the methods we have employed seem to be applicable for solving the problem of reciprocal sum of Fibonacci numbers and its generalization (see [10]), a problem that has attracted many recent researches ([11], [14], [12], [13], etc.).

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Competing Interests

The authors declare that no competing interests exist.

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