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An Innovative Solutions for the Generalized FitzHugh-Nagumo Equation by Using the Generalized $\left(\dfrac{G'}{G}\right)$-Expansion Method

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Abstract

In this paper, the generalized $\left(\dfrac{G'}{G}\right)$-expansion method is used for construct an innovative explicit traveling wave solutions involving parameter of the generalized FitzHugh-Nagumo equation $u_t = u_{xx} - u(1-u)(a(t)-u)$, for some special parameter $a(t)$ where $G = G(\xi)$ satisfies a second order linear differential equation $G'' + \lambda G' + \mu G = 0$, $\xi = p(t)x + q(t)$, where $p(t)$ and $q(t)$ are functions of t.

1. Introduction

Phenomena in physics and other fields are often described by nonlinear evolution equations (NLEEs). When we want to understand the physical mechanism of phenomena in nature, described by nonlinear evolution equations, exact solutions for the nonlinear evolution equations have to be explored. For example, the wave phenomena observed in fluid dynamics [1,2], plasma and elastic media [3,4] and optical fibers [5,6], etc. In the past several decades, many effective methods for obtaining exact solutions of NLEEs have been proposed, such as Hirota's bilinear method [7], Backlund transformation [8], Painlevé expansion [9], sine-cosine method [10], homogeneous balance method [11], homotopy perturbation method [12-14], variational iteration method [15-18], asymptotic methods [19], non-perturbative methods [20], Adomian decomposition method [21], tanh-function method [22-26], algebraic method [27-30]. Jacobi elliptic function expansion method [31-33], F-expansion method [34-36] and auxiliary equation method [37-40]. Recently, Wang *et al.* [41] introduced a new direct method called the $\left(\dfrac{G'}{G}\right)$-expansion method to look for travelling wave solutions of NLEEs. The $\left(\dfrac{G'}{G}\right)$-expansion method is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in $\left(\dfrac{G'}{G}\right)$, and that $G = G(\xi)$ satisfies a second order linear ordinary differential equation (LODE):

$$G'' + \lambda G' + \mu G = 0, \text{ where } G' = \frac{dG(\xi)}{d\xi}, \ G'' = \frac{d^2 G(\xi)}{d\xi^2},$$

$\xi = x - Vt$, V is a constant. The degree of the polynomial can be determined by considering the homogeneous balance between the highest order derivative and nonlinear terms appearing in the given NLEE. The coefficients of the polynomial can be obtained by solving a set of algebraic equations resulted from the process of

using the method. By using the $\left(\dfrac{G'}{G}\right)$-expansion method, Wang *et al.* [41] successfully obtained more travelling wave solutions of four NLEEs. Very recently, Zhang *et al.* [42] proposed a generalized $\left(\dfrac{G'}{G}\right)$-expansion method [42] to improve the work made in [41]. The main purpose of this paper is to use generalized $\left(\dfrac{G'}{G}\right)$-expansion method to solve the generalized FitzHugh-Nagumo equation. The performance of this method is reliable, simple and gives many new solutions, its also standard and computerizable method which enable us to solve complicated nonlinear evolution equations in mathematical physics. The paper is organized as follows. In section 2, we describe briefly the generalized $\left(\dfrac{G'}{G}\right)$-expansion method, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G'' + \lambda G' + \mu G = 0$, $\xi = p(t)x + q(t)$ In section 3, we apply this method to the FitzHugh-Nagumo equation. In section 4, some conclusions are given.

2. Description the Generalized $\left(\dfrac{G'}{G}\right)$-Expansion Method

Suppose that we have the following nonlinear partial differential equation

$$P\left(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \cdots\right) = 0, \tag{2.1}$$

we suppose its solution can be expressed by a polynomial $\left(\dfrac{G'}{G}\right)$ as follows:

$$u(\xi) = \sum_{i=1}^{n} \alpha_i(t)\left(\dfrac{G'}{G}\right)^i + \alpha_0(t), \quad \alpha_j(t) \neq 0, \tag{2.2}$$

where $\alpha_0(t)$ and $\alpha_j(t)$ are functions of t $(j = 1, 2, \cdots, n)$ and $\xi = \xi(x,t)$ is a function of x, t to be determine later, $G = G(\xi)$ satisfies the second order linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{2.3}$$

To determine u explicitly we take the following four steps.

Step 1. Determine the integer n by balancing the highest order nonlinear term(s) and the highest order partial derivative of u in Equation (2.1).

Step 2. Substitute Equation (2.2) along with Equation (2.3) into Equation (2.1) and collect all terms with the same order of $\left(\dfrac{G'}{G}\right)$ together, the left hand side of Equation (2.1) is converted into a polynomial in $\left(\dfrac{G'}{G}\right)$. Then set each coefficient of this polynomial to zero to derive a set of over-determined partial differential equations for $\alpha_0(t)$, $\alpha_i(t)$ and ξ.

Step 3. Solve the system of all equations obtained in step 2 for $\alpha_0(t)$, $\alpha_i(t)$ and ξ by use of Maple.

Step 4. Use the results obtained in above steps to derive a series of fundamental solutions of Equation (2.3) depending on $\left(\dfrac{G'}{G}\right)$, since the solutions of this equation have been well known for us, then we can obtain exact solutions of Equation (2.1).

3. The FitzHugh-Nagumo Equation

In this section, we apply the generalized $\left(\dfrac{G'}{G}\right)$-expansion method to solve the generalized FitzHugh-Nagumo equation, construct the traveling wave solutions for it as follows:

Let us first consider the generalized FitzHugh-Nagumo equation

$$u_t = u_{xx} - u(1-u)\left(a(t)-u\right) \tag{3.1}$$

where $a(t)$ is a function of t. In order to look for the traveling wave solutions of Equation (3.1) we suppose that

$$u(x,t) = u(\xi), \xi = p(t)x + q(t) \tag{3.2}$$

Suppose that the solution of Equation (3.1) can be expressed by a polynomial in $\left(\dfrac{G'}{G}\right)$ as follows

$$u(\xi) = \sum_{i=1}^{n} \alpha_i(t)\left(\dfrac{G'}{G}\right)^i + \alpha_0(t) \tag{3.3}$$

Considering the homogeneous balance between u_{xx} and u^3 in Equation (3.1) we required that $n + 2 = 3n$, then $n = 1$. So we can write Equation (3.3) as

$$u(\xi) = \alpha(t)\left(\dfrac{G'}{G}\right) + \alpha_0(t). \tag{3.4}$$

Substituting Equation (3.4) into Equation (3.1) along with Equation (2.3). We obtain the following equations by comparing coefficients of $\left(\dfrac{G'}{G}\right)$. When $j = 3$ then

$$0 = 2\alpha_1 p^2 - \alpha_1^3. \tag{3.5}$$

We solve the equation by setting $\alpha_1 = \sqrt{2}p$ (we could also set $\alpha_1 = -\sqrt{2}p$). The equation for $j = 2$ is

$$-\alpha_1\left(p'x + q'\right) = 3\alpha_1 p^2\lambda - 3\alpha_0\alpha_1^2 + \alpha_1^2 + \alpha_1^2 a. \quad (3.6)$$

We see from this equation that $p(t)$ must be a constant and then $\alpha_1(t)$ is also constant. Therefore, equation Equation (3.6) simplifies to

$$-\alpha_1 q' = 3\alpha_1 p^2\lambda - 3\alpha_0\alpha_1^2 + \alpha_1^2 + \alpha_1^2 a. \quad (3.7)$$

The equation for $j = 1$ is

$$\begin{aligned}-\alpha_1 q'\lambda = 2\alpha_1 p^2\mu + \alpha_1 p^2\lambda^2 + 2\alpha_0\alpha_1 \\ -3\alpha_0^2\alpha_1 + 2\alpha_0\alpha_1 a - \alpha_1 a.\end{aligned} \quad (3.8)$$

We substitute Equation (3.7) into Equation (3.8) and obtain (after dividing by α_1)

$$\begin{aligned}-3\alpha_1\alpha_0\lambda + \alpha_1\lambda + \alpha_1 a\lambda + 2p^2\lambda^2 - 2\alpha_0 + 3\alpha_0^2 \\ -2\alpha_0 a - 2p^2\mu + a = 0.\end{aligned} \quad (3.9)$$

We solve this equation for a and obtain

$$a(t) = \frac{3\alpha_1\alpha_0\lambda - \alpha_1\lambda - 2p^2\lambda^2 + 2\alpha_0 - 3\alpha_0^2 + 2p^2\mu}{\alpha_1\lambda - 2\alpha_0 + 1}. \quad (3.10)$$

The equation for $j = 0$ is

$$\alpha_0' - \alpha_1 q'\mu = \alpha_1 p^2\lambda\mu - \alpha_0 a + \alpha_0^2 + \alpha_0^2 a - \alpha_0^3. \quad (3.11)$$

If we substitute Equation (3.7) and Equation (3.10) into Equation (3.11) we obtain

$$\begin{aligned}&\alpha_0'\left(\alpha_1\lambda - 2\alpha_0 + 1\right) + \alpha_0^4 - \alpha_0^3\left(2\alpha_1\lambda + 2\right) \\ &+\alpha_0^2\left(3\alpha_1^2\mu + 3\alpha_1\lambda + 1 + 2p^2\lambda^2 - 2p^2\mu\right) \\ &+\alpha_0\left(-3\alpha_1^2\mu - 4\alpha_1 p^2\lambda\mu - 2p^2\lambda^2 + 2p^2\mu - \alpha_1\lambda\right) \\ &+2\alpha_1^2\mu^2 p^2 + \alpha_1^2\mu + 2\alpha_1 p^2\lambda\mu = 0.\end{aligned} \quad (3.12)$$

Now Equation (3.12) is an ordinary differential equation for α_0. Therefore, α_0 must have a special form in order to be a solution of this equation which means that the function $a(t)$ expressed in terms of $\alpha_0(t)$ by Equation (3.10) must also of a special form. This shows that we cannot solve all the equations if $a(t)$ is an arbitrary function.

We can still try to find solutions for some special $a(t)$. For example, we choose

$$p = \frac{1}{\sqrt{2}}, \lambda = 1, \mu = 0.$$

Then $\alpha_1 = 1$ and Equation (3.12) simplifies to

$$\alpha_0' + \frac{3}{2}\alpha_0^2 - \alpha_0 - \frac{1}{2}\alpha_0^3 = 0.$$

One solution is

$$\alpha_0(t) = 1 + \frac{1}{\sqrt{1 + e^t}}.$$

We find $a(t)$ from Equation (3.7) as

$$a(t) = \frac{1}{2} + \frac{3}{2}\sqrt{1 + e^t} \quad (3.13)$$

$$q(t) = -3\arctan h\left(\sqrt{1 + e^t}\right).$$

We choose

$$G(\xi) = 1 + e^{-\xi}.$$

Then

$$u = 1 + \frac{1}{\sqrt{1 + e^t}} - \frac{e^{-\xi}}{1 + e^{-\xi}} \quad (3.14)$$

with

$$\xi = \frac{x}{\sqrt{2}} - 3\arctan h\left(\sqrt{1 + e^t}\right)$$

is a solution of equation Equation (3.1) when $a(t)$ is given by Equation (3.13). Once can check with the computer that u given by Equation (3.14) is really a solution of Equation (3.1). It is shows that this method is powerful in constructing exact solutions of NLEEs.

4. Conclusions

This study shows that the generalized $\left(\dfrac{G'}{G}\right)$ -expansion method is quite efficient and practically will suited for use in finding exact solutions for the problem considered here. New and more general excat solutions with arbitrary function $a(t)$ of the generalized FitzHugh-Nagumo equation are obtained, from which some exponential function solutions are also derived when setting the arbitrary function as special values. We construct an innovative explicit traveling wave solutions involving parameter of the generalized FitzHugh-Nagumo equation.

5. References

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