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A Simplified Variant of Chess for which a Short Program Computes a Non-trivial Upper Bound for the Number of Reachable Positions Obtained from 65141475298198504104226577310812726424-036 Naturally Defined Initial Configurations of Pieces

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Short Research Article

Abstract

We simplify the rules of chess. We assume that the initial configuration of pieces is not fixed and satisfies some general conditions. Let $\mathcal{I}$ denote the set of all these configurations. By our assumptions, for every $\mathcal{C} \in \mathcal{I}$, after 0 or more moves, the configuration obtained from $\mathcal{C}$ and the information who has a move determine the set of all ways of continuing the game i.e. the reachable position. For $\mathcal{C} \in \mathcal{I}$, let $\mathcal{R}(\mathcal{C})$ denote the set of all reachable positions obtained from $\mathcal{C}$.

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A short program shows that $\mathrm{card}(\mathcal{I}) = 65141475298198504104226577310812726424036$ and $\mathrm{card}\left(\bigcup_{\mathcal{C}\in\mathcal{I}} \mathcal{R}(\mathcal{C})\right) < 4295923212088255192398899494807384799479217319544$.

Keywords: *Chess in which there are 65141475298198504104226577310812726424036 naturally defined initial configurations of pieces; chess with simplified rules; upper bound for the number of reachable positions.*

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1 Zermelo's Theorem and a Rough Upper Bound for the Number of Reachable Positions in the Classical Chess

The rules of the classical chess are discussed in [1], [2], [3], and [4].

Theorem 1. *(Zermelo's theorem, [5, p. 128]). A player who is in a winning position is always able to enforce a win in a number of moves that is less than the number of positions in the game.*

Lemma 1. *(cf. [6]). The number of configurations of the pieces on the chessboard does not exceed* 13^{64}.

Proof. The number 13 corresponds to 12 distinct pieces and the empty square. □

Theorem 2. *(cf. [6]). The number of reachable positions in chess does not exceed* $13^{64} \cdot 2 \cdot 9 \cdot (4 \cdot 4)$.

Proof. It follows from Lemma 1 and the text below. The number 2 in the formula corresponds to the alternative:

$$(\texttt{White}\ \textit{has a move}) \vee (\texttt{Black}\ \textit{has a move})$$

A legality of a move depends not only on the configuration of the pieces on the chessboard due to en passant captures and castlings. We number the squares of the chessboard from 1 to 64. This numbers white (black) pawns starting with 1 and ending with some $\alpha \in \{0, \ldots, 8\}$. The en passant capture is only possible when some opposing pawn has advanced two squares on the previous move. At most one opposing pawn satisfies this condition. The number 9 in the formula corresponds to the alternative

$$(\textit{the opposing pawn } 1 \textit{ has advanced two squares on the previous move}) \vee \ldots \vee$$

$$(\textit{the opposing pawn } \alpha \textit{ has advanced two squares on the previous move}) \vee$$
$$(\textit{no opposing pawn has made such a previous move})$$

which holds for both players. The number $4 \cdot 4$ in the formula corresponds to the alternative

$$(\textit{castlings are not possible for us}) \vee (\textit{only long castling is possible for us}) \vee$$

$$(\textit{only short castling is possible for us}) \vee (\textit{both castlings are possible for us})$$

which holds for both players. □

Non-trivial upper bounds for the number of reachable positions are discussed in [7], [6], [8], and [9].

2 A Simplified Variant of Chess with a Non-fixed Initial Configuration of Pieces

To simplify the rules of chess we assume: only checkmate or stalemate ends the chess game, there are no castlings and en passant captures, no pawn can advance two squares in one move. We assume that the initial configuration of pieces is not fixed and satisfies the following conditions:

(1) For every colour, the number of pawns belongs to the set $\{0, \ldots, 8\}$.
(2) Pawns are in rows $2-7$.
(3) For every colour, the number of queens belongs to the set $\{0, 1\}$.
(4) For every colour, the number of bishops (knights, rooks) belongs to the set $\{0, 1, 2\}$.
(5) The kings are not in adjacent squares.

3 42959232120882551923988994948073848799479217319544 is Greater than the Number of Reachable Positions Obtained from 65141475298198504104226577310812726424036 Naturally Defined Initial Configurations of Pieces

Let $\mathcal{I}$ denote the set of all initial configurations of pieces satisfying conditions (1)-(5). By our assumptions, for every $\mathcal{C} \in \mathcal{I}$, after 0 or more moves, the configuration obtained from $\mathcal{C}$ and the information who has a move determine the set of all ways of continuing the game i.e. the reachable position. For $\mathcal{C} \in \mathcal{I}$, let $\mathcal{R}(\mathcal{C})$ denote the set of all reachable positions obtained from $\mathcal{C}$.

For every $\mathcal{C} \in \mathcal{I}$, the bound from Theorem 3 applies to Theorem 1.

Theorem 3.

$$\mathrm{card}(\mathcal{I}) = 65141475298198504104226577310812726424036$$

$$\mathrm{card}\left(\bigcup_{\mathcal{C} \in \mathcal{I}} \mathcal{R}(\mathcal{C})\right) \leqslant 42959232120882551923988994948073848799479217319544$$

Proof. The execution of the following *MuPAD* program returns the above numbers.

```
kings0:=0:
/* kings0 equals the number of legal configurations of kings */
/* in which none king is in rows 2-7 */
kings1:=0:
/* kings1 equals the number of legal configurations of kings */
/* in which exactly one king is in rows 2-7 */
kings2:=0:
/* kings2 equals the number of legal configurations of kings */
/* in which both kings are in rows 2-7 */
for king1x from 1 to 8 do
for king1y from 1 to 8 do
for king2x from 1 to 8 do
for king2y from 1 to 8 do
m:=0:
if king1y>1 and king1y<8 then m:=m+1 end_if:
if king2y>1 and king2y<8 then m:=m+1 end_if:
if abs(king1x-king2x)<2 and abs(king1y-king2y)<2 then m:=3 end_if:
if m=0 then kings0:=kings0+1 end_if:
if m=1 then kings1:=kings1+1 end_if:
if m=2 then kings2:=kings2+1 end_if:
end_for:
end_for:
```

```
end_for:
end_for:
initialconfigurations:=0:
bound:=0:
for n from 0 to 2 do
if n=0 then ki:=kings0 end_if:
if n=1 then ki:=kings1 end_if:
if n=2 then ki:=kings2 end_if:
for p1 from 0 to 8 do
/* p1 denotes the number of white pawns */
pa1:=binomial(64-16-n,p1):
for p2 from 0 to 8 do
/* p2 denotes the number of black pawns */
pa2:=binomial(64-16-n-p1,p2):
for q1 from 0 to 1+8-p1 do
/* q1 denotes the number of white queens */
qu1:=binomial(64-2-p1-p2,q1):
for q2 from 0 to 1+8-p2 do
/* q2 denotes the number of black queens */
qu2:=binomial(64-2-p1-p2-q1,q2):
for b1 from 0 to 2+8-p1-max(q1-1,0) do
/* b1 denotes the number of white bishops */
bi1:=binomial(64-2-p1-p2-q1-q2,b1):
for b2 from 0 to 2+8-p2-max(q2-1,0) do
/* b2 denotes the number of black bishops */
bi2:=binomial(64-2-p1-p2-q1-q2-b1,b2):
for k1 from 0 to 2+8-p1-max(q1-1,0)-max(b1-2,0) do
/* k1 denotes the number of white knights */
kn1:=binomial(64-2-p1-p2-q1-q2-b1-b2,k1):
for k2 from 0 to 2+8-p2-max(q2-1,0)-max(b2-2,0) do
/* k2 denotes the number of black knights */
kn2:=binomial(64-2-p1-p2-q1-q2-b1-b2-k1,k2):
for r1 from 0 to 2+8-p1-max(q1-1,0)-max(b1-2,0)-max(k1-2,0) do
/* r1 denotes the number of white rooks */
ro1:=binomial(64-2-p1-p2-q1-q2-b1-b2-k1-k2,r1):
for r2 from 0 to 2+8-p2-max(q2-1,0)-max(b2-2,0)-max(k2-2,0) do
/* r2 denotes the number of black rooks */
ro2:=binomial(64-2-p1-p2-q1-q2-b1-b2-k1-k2-r1,r2):
d:=ki*pa1*pa2*qu1*qu2*bi1*bi2*kn1*kn2*ro1*ro2:
a:=d:
if q1>1 or q2>1 or b1>2 or b2>2 or k1>2 or k2>2 or r1>2 or r2>2
then a:=0 end_if:
initialconfigurations:=initialconfigurations+a:
bound:=bound+d:
end_for:
end_for:
end_for:
```

```
end_for:
end_for:
end_for:
end_for:
end_for:
end_for:
end_for:
end_for:
print(initialconfigurations):
print(2*bound):
/* the factor 2 corresponds to the alternative */
/* (White has a move) or (Black has a move) */
```

Pawns cannot be in rows 1 and 8. It applies to the instructions with the numbers $64-16-n$ and $64-16-n-p1$. Non-negative integers $p1$, $p2$, $q1$, $q2$, $b1$, $b2$, $k1$, $k2$, $r1$, $r2$ satisfy:
$p1, p2 \in \{0, \dots, 8\}$
$q1 \in \{0, \dots, 1+8-p1\}$
$q2 \in \{0, \dots, 1+8-p2\}$
$b1 \in \{0, \dots, 2+8-p1-\max(q1-1,0)\}$
$b2 \in \{0, \dots, 2+8-p2-\max(q2-1,0)\}$
$k1 \in \{0, \dots, 2+8-p1-\max(q1-1,0)-\max(b1-2,0)\}$
$k2 \in \{0, \dots, 2+8-p2-\max(q2-1,0)-\max(b2-2,0)\}$
$r1 \in \{0, \dots, 2+8-p1-\max(q1-1,0)-\max(b1-2,0)-\max(k1-2,0)\}$
$r2 \in \{0, \dots, 2+8-p2-\max(q2-1,0)-\max(b2-2,0)-\max(k2-2,0)\}$

We have: `bound =`

$$\sum_{\substack{n\in\{0,1,2\}\\ p1,p2\in\{0,\dots,8\}\\ q1,q2\in\{0,\dots,1+8\}\\ b1,b2,k1,k2,r1,r2\in\{0,\dots,2+8\}\\ p1+\max(q1-1,0)+\max(b1-2,0)+\\ \max(k1-2,0)+\max(r1-2,0)\leqslant 8\\ p2+\max(q2-1,0)+\max(b2-2,0)+\\ \max(k2-2,0)+\max(r2-2,0)\leqslant 8}} ki\cdot pa1\cdot pa2\cdot qu1\cdot qu2\cdot bi1\cdot bi2\cdot kn1\cdot kn2\cdot ro1\cdot ro2$$

The *MuPAD* program uses the above equality. □

Theorem 4. *The inequality in Theorem 3 is strong.*

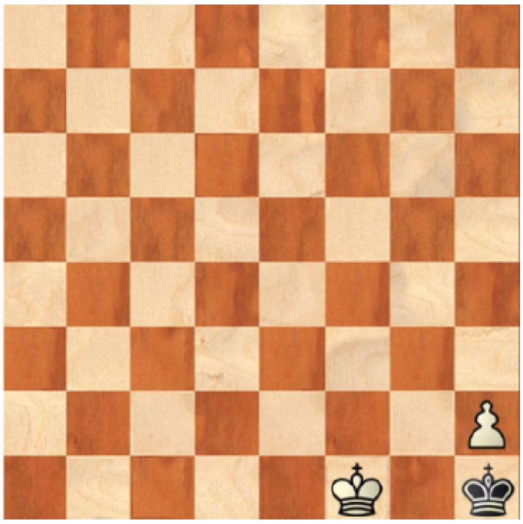
Proof. The bound in Theorem 3 takes into account many cases such that there are at least two queens of the same colour and black king is in check whereas `White` has a move. Every such case is not reachable because every configuration from $\mathcal{I}$ contains at most one white (black) queen. □

4 No Single Configuration from $\mathcal{I}$ Produces the All Reachable Positions Obtained from the All Configurations from $\mathcal{I}$

Theorem 5. *For every* $\mathcal{C}_0 \in \mathcal{I}$, $\mathcal{R}(\mathcal{C}_0) \subsetneq \bigcup\limits_{\mathcal{C}\in\mathcal{I}} \mathcal{R}(\mathcal{C})$.

Proof. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ denote the following elements of $\mathcal{I}$:

Configuration $\mathcal{C}_1$



Configuration $\mathcal{C}_2$

We have:

$$\forall \mathcal{C} \in \mathcal{I} \;\; (\mathcal{C}_1 \in \mathcal{R}(\mathcal{C}) \Rightarrow \mathcal{C} = \mathcal{C}_1)$$

$$\forall \mathcal{C} \in \mathcal{I} \;\; (\mathcal{C}_2 \in \mathcal{R}(\mathcal{C}) \Rightarrow \mathcal{C} = \mathcal{C}_2)$$

Assume, on the contrary, that there exists $\mathcal{C}_0 \in \mathcal{I}$ such that $\mathcal{R}(\mathcal{C}_0) = \bigcup\limits_{\mathcal{C} \in \mathcal{I}} \mathcal{R}(\mathcal{C})$. We have:

$$\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{I} \subseteq \bigcup_{\mathcal{C} \in \mathcal{I}} \mathcal{R}(\mathcal{C}) = \mathcal{R}(\mathcal{C}_0)$$

Hence, $\mathcal{C}_0 = \mathcal{C}_1$ and $\mathcal{C}_0 = \mathcal{C}_2$. We get a contradiction because $\mathcal{C}_1 \neq \mathcal{C}_2$. □

Competing Interests

Author has declared that no competing interests exist.

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