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Research Article

Klein-Gordon Oscillator in the Presence of External Fields in a Cosmic Space-Time with a Space-Like Dislocation and Aharonov-Bohm Effect

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In this paper, we study interactions of a scalar particle with electromagnetic potential in the background space-time generated by a cosmic string with a space-like dislocation. We solve the Klein-Gordon oscillator in the presence of external fields including an internal magnetic flux field and analyze the analogue effect to the Aharonov-Bohm effect for bound states. We extend this analysis subject to a Cornell-type scalar potential and observe the effects on the relativistic energy eigenvalue and eigenfunction.

1. Introduction

The Klein-Gordon oscillator [1, 2] was inspired by the Dirac oscillator [3] applied to spin-(1/2) particles. The Klein-Gordon oscillator has been investigated in several physical systems, such as in the background of the cosmic string with external fields [4], in the presence of a Coulomb-type potential considering two ways: (i) by modifying the mass term $m \longrightarrow m + S$ [5] and (ii) via the minimal coupling [6] with a linear potential, in the background space-time produced by topological defects using the Kaluza-Klein theory [7], in the Som-Raychaudhuri space-time in the presence of external fields [8], in the motion of an electron in an external magnetic field in the presence of screw dislocations [9], in the continuous distribution of screw dislocation [10], in the presence of a Cornell-type potential in a cosmic string space-time [11], in the relativistic quantum dynamics of a DKP oscillator field subject to a linear scalar potential [12], in the DKP equation for spin-zero bosons subject to a linear scalar potential [13], and in the Dirac equation subject to a vector and scalar potentials [14]. In the literature, it is known that a cosmic string has been produced by phase transitions in the early universe [15] as it is predicted in the extensions of the standard model [16, 17]. Topological defects in condensed matter physics can be associated with the presence of curvature and torsion. In particular, topological effects associated with torsion have been investigated in crystalline solids with the use of differential geometry [18, 19]. Recent studies have explored the effects of torsion on condensed matter systems [20–23]. Therefore, there is a great interest in the connection between quantum mechanics and the general relativity.

The cosmic string space-time in cylindrical coordinates (t, r, ϕ, z) is described by the following line element [16–18, 24–29]:

$$ds^2 = -dt^2 + dr^2 + \alpha^2 r^2 d\phi^2 + dz^2, \quad (1)$$

where $\alpha = (1 - 4\mu)$ is the topological parameter with μ being the linear mass density of a cosmic string and $0 < \alpha < 1$. Furthermore, in the cylindrical symmetry, we have that $0 < r \leq \infty$, $0 \leq \phi \leq 2\pi$, and $-\infty < z < \infty$.

In Ref. [30], the Klein-Gordon oscillator without and/or with a linear scalar potential in the presence of external fields including an internal magnetic flux field in a space-time with a space-like dislocation was studied. They solved the wave equation analytically and analyzed the effects on the relativistic energy eigenvalue. In Ref. [31], authors investigated the Klein-Gordon oscillator with topological defects including an internal magnetic flux field subject to a Coulomb-type plus linear potential (called a Cornell-type potential) in a space-time with screw dislocation (space-like dislocation). They obtained the relativistic energy eigenvalue and observed the analogous effect to the Aharonov-Bohm effect for bound states. In this work, we study the Klein-Gordon oscillator field interaction with external fields including an internal magnetic flux field in a space-time with a magnetic screw dislocation. We extend this analysis subject to a Cornell-type scalar potential by searching analytically for a bound state solution to the system. Recently, dislocation has been investigated in nonrelativistic and relativistic quantum systems. The spiral dislocation, in the nonrelativistic context, has been investigated in the harmonic oscillator [32]; in the relativistic context, it has been investigated in a scalar field in a noninertial frame [33]. The screw dislocation, in the nonrelativistic context, has been applied in the harmonic oscillator [34, 35], in the Landau quantization [9, 10, 36], in the doubly anharmonic oscillator [37], in Landau quantization for an induced electric dipole [38], and in noninertial effects on a nonrelativistic Dirac particle [39]. In the relativistic context, the screw dislocation has been studied in the Dirac oscillator [40, 41], in the Klein-Gordon oscillator [31], and in the analogue effects to the Aharonov-Bohm effect for bound states in a position-dependent mass system [42].

The structure of the present paper is as follows: in Section 2.1, we investigate the Klein-Gordon oscillator in the presence of an external field including an internal magnetic flux in a cosmic string space-time with a space-like dislocation; in Section 2.2, we extend this analysis subject to a Cornell-type scalar potential and analyze the analogue effect to the Aharonov-Bohm effect for bound states; and finally, conclusions are presented in Section 3.

2. Klein-Gordon Oscillator Interacts with External Fields in a Cosmic Space-Time with a Space-Like Dislocation

Let us begin this section by introducing the space-time with a screw dislocation. It corresponds to a space-time with a linear topological defect associated with torsion and a cosmic string that can be described by the line element [30]

$$ds^2 = -dt^2 + dr^2 + \alpha^2 r^2 \, d\phi^2 + (dz + \chi \, d\phi)^2, \qquad (2)$$

where $c = 1 = \hbar$, $0 < \alpha < 1$ is the cosmic string parameter, and χ is the dislocation (torsion) parameter, and in condensed matter physics, this parameter is related to the Burgers vector $\mathbf{b}$ via $\chi = (b/2\pi)$ [18, 19, 43, 44]. It is important to mention that the screw dislocation (torsion) corresponds to a singularity at the origin [18, 43]. Also note that the spatial part of the metric (2) is called the Katanaev and Volovich line element in studies of solids as the screw dislocation [18, 19]. For $\chi \longrightarrow 0$, the metric (2) reduces to a cosmic string space-time. Again for $\chi \longrightarrow 0$, and $\alpha \longrightarrow 1$, the space-time reduces to the Minkowski flat space metric in cylindrical coordinate system.

The metric tensor for the space-time (2) is

$$g_{\mu\nu}(x) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha^2 r^2 + \chi^2 & \chi \\ 0 & 0 & \chi & 1 \end{pmatrix}, \qquad (3)$$

with its inverse

$$g^{\mu\nu}(x) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha^2 r^2} & -\frac{\chi}{\alpha^2 r^2} \\ 0 & 0 & -\frac{\chi}{\alpha^2 r^2} & 1 + \frac{\chi^2}{\alpha^2 r^2} \end{pmatrix}. \qquad (4)$$

The metric has the signature $(-, +, +, +)$, and the determinant of the corresponding metric tensor $g_{\mu\nu}$ is

$$\det \, g = -\alpha^2 r^2. \qquad (5)$$

The relativistic quantum dynamics of charged particles of mass m is described by the Klein-Gordon equation [28]

$$\left[\frac{1}{\sqrt{-g}} D_\mu \left(\sqrt{-g} g^{\mu\nu} D_\nu\right) - m^2\right] \Psi = 0, \qquad (6)$$

where the minimal coupling with electromagnetic interaction is as follows:

$$D_\mu = \partial_\mu - ieA_\mu, \qquad (7)$$

where e is the electric charge and A_μ is the electromagnetic four-vector potential defined by

$$A_\mu = \left(0, \vec{A}\right), \vec{A} = \left(0, A_\phi, 0\right). \qquad (8)$$

The three-vector potential in the symmetric gauge is defined by

$$\vec{A} = \vec{A}_1 + \vec{A}_2, \qquad (9)$$

where the angular component of three-vector potential for $\vec{A}_1$ is [28].

$$A_\phi^{(1)} = -\frac{1}{2} \alpha B_0 r^2. \qquad (10)$$

And that for $\vec{A}_2$ is [30, 31, 42, 45, 46]

$$A_\phi^{(2)} = \frac{\Phi_B}{2\pi}, \qquad (11)$$

so that the angular component of the electromagnetic four-vector potential is

$$A_\phi = -\frac{1}{2}\alpha B_0 r^2 + \frac{\Phi_B}{2\pi}. \tag{12}$$

The magnetic field is along the z-direction given by

$$\vec{B} = \vec{\nabla} \times \vec{A} = -B_0 \hat{k}. \tag{13}$$

Here $\Phi_B = \text{const}$ is an internal quantum magnetic flux through the core of the topological defect [34] and $eA_\phi = \Phi$ where $\Phi = (\Phi_B/(2\pi/e))$.

If one introduces a scalar potential by modifying the mass term as $m \longrightarrow m + S(r)$ into the above equation, we have

$$\left[\frac{1}{\sqrt{-g}} D_\mu \left(\sqrt{-g} g^{\mu\nu} D_\nu\right) - (m+S)^2\right]\Psi = 0. \tag{14}$$

Using the geometry (2), Equation (14) becomes

$$\left[-\partial_t^2 + \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{\alpha^2 r^2}\left(\partial_\phi - ieA_\phi - \chi\partial_z\right)^2 + \partial_z^2 - (m+S)^2\right]\Psi = 0. \tag{15}$$

To include the oscillator with the Klein-Gordon field, we change the following momentum operator [7]:

$$\vec{p} \to \vec{p} + im\Omega\vec{r}, \tag{16}$$

where Ω is the oscillator frequency and $\vec{r} = r\hat{r}$ with r being the distance from the particle to the string. So we can write $p^2 \longrightarrow (\vec{p} + im\Omega\vec{r})(\vec{p} - im\Omega\vec{r})$. Therefore, Equation (15) becomes

$$\begin{aligned}&\left[-\partial_t^2 + \frac{1}{r}\left(\partial_r + m\Omega r\right)\left(r\partial_r - m\Omega r^2\right) + \frac{1}{\alpha^2 r^2}\left(\partial_\phi - ieA_\phi - \chi\partial_z\right)^2 + \partial_z^2 - (m+S)^2\right]\Psi \\ &= 0 \Rightarrow \left[-\partial_t^2 + \frac{1}{r}\partial_r(r\partial_r) - 2m\Omega - m^2\Omega^2 r^2 + \frac{1}{\alpha^2 r^2}\left(\partial_\phi - ieA_\phi - \chi\partial_z\right)^2 + \partial_z^2 - (m+S)^2\right]\Psi = 0.\end{aligned} \tag{17}$$

We choose the following ansatz for the function Ψ:

$$\Psi(t, r, \phi, z) = e^{i(-Et + l\phi + kz)}\psi(r), \tag{18}$$

where E is the energy, $l = 0, \pm 1, \pm 2, \cdots \in \mathbf{Z}$ are the eigenvalue of the z-component of the angular momentum operator, and k is a constant.

Substituting the above ansatz (18) into Equation (17), we have

$$\psi''(r) + \frac{1}{r}\psi'(r) + \left[E^2 - 2m\Omega - m^2\Omega^2 r^2 - \frac{1}{\alpha^2 r^2}\left(l - eA_\phi - k\chi\right)^2 - k^2 - (m+S)^2\right]\psi = 0. \tag{19}$$

2.1. Interaction without a Scalar Potential $S = 0$.

Here, we investigate the gravitational effect produced by the topological defects (cosmic string) on the above relativistic quantum system without scalar potential in the presence of external fields including an internal magnetic flux field. We see that the relativistic energy eigenvalue is modified by the topological defects and break their degeneracy.

Substituting the angular component of the four-vector potential Equation (12) into Equation (19), we obtain the following differential equation:

$$\psi''(r) + \frac{1}{r}\psi'(r) + \left[\lambda - m^2\omega^2 r^2 - \frac{j^2}{r^2}\right]\psi = 0, \tag{20}$$

where

$$\begin{aligned}\lambda &= E^2 - k^2 - \frac{2m\omega_c}{\alpha}\left(\alpha l_{\text{eff}} - k\chi\right) - m^2 - 2m\Omega, \\ j &= \frac{|l - \Phi - k\chi|}{\alpha}, \\ \omega &= \sqrt{\omega_c^2 + \Omega^2}, \\ l_{\text{eff}} &= \frac{1}{\alpha}(l - \Phi), \\ \Phi &= \frac{\Phi_B}{2\pi/e},\end{aligned} \tag{21}$$

and the cyclotron frequency is

$$\omega_c = \frac{eB_0}{2m}. \tag{22}$$

Transforming $s = m\omega r^2$ into the above equation, we obtain the following differential equation [47]:

$$\psi''(s) + \frac{1}{s}\psi'(s) + \frac{1}{s^2}\left(-\xi_1 s^2 + \xi_2 s - \xi_3\right)\psi(s) = 0, \tag{23}$$

where

$$\begin{aligned}\xi_1 &= \frac{1}{4}, \\ \xi_2 &= \frac{\lambda}{4m\omega}, \\ \xi_3 &= \frac{j^2}{4}.\end{aligned} \tag{24}$$

The energy eigenvalue is given by

$$\begin{aligned}(2n+1)\sqrt{\xi_1}-\xi_2+2\sqrt{\xi_1\xi_3}\\ &=0\Rightarrow\lambda=2m\sqrt{\omega_c^2+\Omega^2}(2n+1+j)\Rightarrow E_{n,l}^2\\ &=m^2+k^2+2m\Omega+2m\sqrt{\omega_c^2+\Omega^2}\left(2n+1+\frac{|l-k\chi-\Phi|}{\alpha}\right)\\ &\quad+2m\omega_c\frac{l-k\chi-\Phi}{\alpha},\end{aligned} \tag{25}$$

where $n=0,1,2,3,4,\cdots$.

The eigenfunction is given by

$$\psi(s)=s^{(|l-\Phi-k\chi|)/2\alpha}e^{-(s/2)}L_n^{(|l-\Phi-k\chi|)/\alpha}(s), \tag{26}$$

where $L_n^{(\beta)}(s)$ is the generalized Laguerre polynomials.

If we take $\alpha\longrightarrow 1$, one will recover the results obtained in [30] (see Equation (31) in [30]). As the cosmic string parameter$\alpha$is composed of the values$0<\alpha<1$, we can see that the presence of the cosmic string parameterαmodifies the energy spectrum. For $\Phi_B\neq 0$ and $\chi\neq 0$, we can observe in Equation (25) that there exists an effective angular momentum, $l\longrightarrow l_{\text{eff}}=(1/\alpha)(l-\Phi-k\chi)$. Thus, the relativistic energy eigenvalue depends on the Aharonov-Bohm geometric quantum phase [48]. This dependence of the energy eigenvalue on the geometric quantum phase gives rise to the analogous effect to the Aharonov-Bohm effect for bound states [49–51]. Besides, we have that $E_{n,\bar{l}}(\Phi_B+\Phi_0)=E_{n,\bar{l}\mp\tau}(\Phi_B)$ where $\Phi_0=\pm(2\pi\alpha/e)\tau$ with $\tau=1,2,3,\cdots$ and $\bar{l}=(l/\alpha)$.

For the zero torsion parameter, $\chi=0$, Equation (25) becomes

$$\begin{aligned}E_{n,l}^2&=m^2+k^2+2m\Omega\\ &\quad+2m\sqrt{\omega_c^2+\Omega^2}\left(2n+1+\frac{|l-\Phi|}{\alpha}\right)+2m\omega_c\frac{l-\Phi}{\alpha}.\end{aligned} \tag{27}$$

Equation (27) is the energy spectrum of the Klein-Gordon oscillator in the presence of an external uniform magnetic field including an internal magnetic flux in a cosmic string space-time. For $\Phi_B\neq 0$, we can observe in Equation (27) that the angular quantum number is shifted, $l\longrightarrow l'=(1/\alpha)(l-(\phi_B/(2\pi/e)))$, and thus, the relativistic energy eigenvalue depends on the Aharonov-Bohm geometric phase [48]. This dependence of the energy eigenvalue on the geometric quantum phase gives rise to the analogous effect to the Aharonov-Bohm effect for bound states [49–51]. Besides, we have that $E_{n,\bar{l}}(\Phi_B+\Phi_0)=E_{n,\bar{l}\mp\tau}(\Phi_B)$, where $\Phi_0=\pm(2\pi\alpha/e)\tau$ with $\tau=1,2,3,\cdots$ and $\bar{l}=l/\alpha$. By taking $\Phi_B=0$ in Equation (27), we have that the relativistic energy levels arise from the interaction of the Klein-Gordon oscillator with a uniform magnetic field in the Minkowski space-time with a cosmic string. On the other hand, by taking $\omega_c\longrightarrow 0$ and $\Phi_B\longrightarrow 0$ in Equation (27), we recover the results obtained in Ref. [7]. Thus, for $\chi\neq 0$ and $\Phi_B\neq 0$ in the energy eigenvalue Equation (25), we have that the presence of torsion in the space-time modifies the degeneracy of the relativistic energy levels. Besides, the presence of torsion in the space-time changes the pattern of oscillation of the energy levels.

2.2. Interaction with a Cornell-Type Scalar Potential. Here, we investigate the above relativistic quantum system described by the Klein-Gordon oscillator subject to a Cornell-type scalar potential in the presence of external fields including an internal magnetic flux field. A scalar potential is included into the systems by modifying the mass $m\longrightarrow m+S(r)$ which is called a position-dependent mass system in the relativistic quantum systems (see, e.g., [5, 6, 8, 28, 30, 31, 42, 46, 52–62]).

The Cornell potential, which consists of a linear potential plus a Coulomb potential, is a particular case of the quark-antiquark interaction, which has one more harmonic type term [53]. Recently, the Cornell potential has been studied in the ground state of three quarks [63]. However, this type of potential is worked on spherical symmetry; in cylindrical symmetry, which is in our case, this type of potential is known as a Cornell-type potential [8, 31, 54–56].

The Cornell-type scalar potential is given by

$$S=\frac{\eta_c}{r}+\eta_L r, \tag{28}$$

where η_c and η_L are the positive arbitrary potential parameters. This potential has been used successfully in models describing binding states of heavy quarks [64–66]. The Cornell potential contains a short-range part dominated by a Coulombic term of quark and gluon interaction $\sim(a/r)$ and the large distance quark confinement as a linear term $\sim br$ [67–72]. In some situations when the parameter b is small, it provides a particular case of perturbed Coulomb problem in atomic physics [73]. This potential has been used to study the strange, charmed, and beautiful baryon masses in the framework of a variational approach [74].

Substituting the vector potential Equation (12) and the scalar potential Equation (28) into Equation (19), we obtain the following differential equation:

$$\psi''(r)+\frac{1}{r}\psi'(r)+\left[\tilde{\lambda}-\tilde{\omega}^2r^2-\frac{\tilde{j}^2}{r^2}-\frac{a}{r}-br\right]\psi(r)=0, \tag{29}$$

where

$$\begin{aligned}\tilde{\lambda}&=E^2-m^2-k^2-2\eta_c\eta_L-\frac{2m\omega_c}{\alpha}(l-\Phi-k\chi)-2m\Omega,\\ \tilde{\omega}&=\sqrt{m^2\left(\omega_c^2+\Omega^2\right)+\eta_L^2},\\ \tilde{j}&=\sqrt{\frac{(l-\Phi-k\chi)^2}{\alpha^2}+\eta_c^2},\\ a&=2m\eta_c,\\ b&=2m\eta_L.\end{aligned} \tag{30}$$

Transforming $x = \sqrt{\tilde{\omega}} r$ into the above Equation (29), we obtain the following wave equation:

$$\psi''(x) + \frac{1}{x}\psi'(x) + \left[\zeta - x^2 - \frac{\tilde{j}^2}{x^2} - \frac{\eta}{x} - \theta x\right]\psi(x) = 0, \quad (31)$$

where we have defined

$$\zeta = \frac{\tilde{\lambda}}{\tilde{\omega}}, \quad \eta = \frac{a}{\sqrt{\tilde{\omega}}}, \quad \theta = \frac{b}{\tilde{\omega}^{3/2}}. \quad (32)$$

Now, we use the appropriate boundary conditions to investigate the bound state solution in this problem. It is required that the wave functions must be regular both at $x \longrightarrow 0$ and $x \longrightarrow \infty$. These conditions are necessary since the wave functions must be well-behaved in these limits, and thus, bound states of energy for the system can be obtained. Suppose the possible solution to Equation (31) is

$$\psi(x) = x^{\tilde{j}} e^{-(1/2)(\theta + x)x} H(x), \quad (33)$$

where $H(x)$ is an unknown function. Substituting solution (33) into Equation (31), we obtain

$$H''(x) + \left[\frac{\gamma}{x} - \theta - 2x\right]H'(x) + \left[-\frac{\beta}{x} + \Theta\right]H(x) = 0, \quad (34)$$

where

$$\gamma = 1 + 2\tilde{j}, \quad \Theta = \zeta + \frac{\theta^2}{4} - 2\left(1 + \tilde{j}\right), \quad \beta = \eta + \frac{\theta}{2}\left(1 + 2\tilde{j}\right). \quad (35)$$

Equation (34) is the biconfluent Heun's differential equation [28, 75, 76] with $H(x)$ as Heun's polynomial function. Many authors studied the analytical solutions to the relativistic wave equations in terms of Heun functions (e.g., [77–82]).

The above Equation (34) can be solved by the series solution method. Writing the solution as a power series expansion around the origin [83],

$$H(x) = \sum_{i=0}^{\infty} c_i x^i. \quad (36)$$

Substituting the power series solution (36) into Equation (34), we get the following recurrence relation for the coefficients:

$$c_{n+2} = \frac{1}{(n+2)\left(n+2+2\tilde{j}\right)}\left[\{\beta + \theta(n+1)\}c_{n+1} - (\Theta - 2n)c_n\right]. \quad (37)$$

And the various coefficients are

$$c_1 = \left(\frac{\eta}{1+2\tilde{j}} + \frac{\theta}{2}\right)c_0, \quad c_2 = \frac{1}{4\left(1+\tilde{j}\right)}\left[(\beta + \theta)\, c_1 - \Theta c_0\right]. \quad (38)$$

As the function $H(x)$ has a power series expansion around the origin in Equation (36), then the relativistic bound state solution can be achieved by imposing that the power series expansion becomes a polynomial of degree n. Through the recurrence relation (37), we can see that the power series expansion $H(r)$ becomes a polynomial of degree n by imposing the following two conditions [28]:

$$\Theta = 2n, \quad c_{n+1} = 0, \quad n = 1, 2, 3, 4, \cdots. \quad (39)$$

By analyzing the condition $\Theta = 2n$, we get the following equation of eigenvalue $E_{n,l}$:

$$E_{n,l}^2 = m^2 + k^2 + 2\eta_c\eta_L + 2m\Omega + \frac{2m\omega_c}{\alpha}(l - k\chi - \Phi) + 2\tilde{\omega}\left(n + 1 + \sqrt{\frac{(l - k\chi - \Phi)^2}{\alpha^2} + \eta_c^2}\right) - \frac{m^2\eta_L^2}{\tilde{\omega}^2}, \quad (40)$$

where $\tilde{\omega}$ is given in Equation (30).

For $\Phi_B \neq 0$ and $\chi \neq 0$, we can observe in Equation (40) that there exists an effective angular momentum quantum number, $l_{\text{eff}} = (1/\alpha)(l - k\chi - (\phi_B/(2\pi/e)))$. Thus, the relativistic energy eigenvalue depends on the Aharonov-Bohm geometric phase [48]. This dependence on the geometric quantum phase gives rise to the analogous effect to the Aharonov-Bohm effect for bound states [49–51]. Besides, we have that $E_{n,\bar{l}}(\Phi_B + \Phi_0) = E_{n,\bar{l}\mp\tau}(\Phi_B)$ where $\Phi_0 = \pm(2\pi\alpha/e)\tau$ with $\tau = 1, 2, 3, \cdots$ and $\bar{l} = l/\alpha$.

The wave function is given by

$$\psi_{n,l}(x) = x^{\sqrt{\left((l - \Phi - k\chi)^2/\alpha^2\right) + \eta_c^2}} e^{-(1/2)(\theta + x)x} H(x). \quad (41)$$

Now, we impose the additional recurrence condition $c_{n+1} = 0$ to find the individual energy levels and corresponding wave functions one by one as done in [84, 85]. As example, for $n = 1$, we have $c_2 = 0$ which implies from (38)

$$c_1 = \frac{2}{\beta + \theta}c_0 \Rightarrow \frac{\eta}{1 + 2\tilde{j}} + \frac{\theta}{2} = \frac{2}{\beta + \theta}, \quad (42)$$

a constraint on the physical parameters from which one can find $\tilde{\omega}_{1,l}$.

Therefore, the ground state energy level for $n = 1$ from (40) is given by

$$E_{1,l}^2 = m^2 + k^2 + 2\eta_c\eta_L + 2m\Omega + \frac{2m\omega_c}{\alpha}(l - k\chi - \Phi) + 2\tilde{\omega}_{1,l}\left(2 + \sqrt{\frac{(l - k\chi - \Phi)^2}{\alpha^2} + \eta_c^2}\right) - \frac{m^2\eta_L^2}{\tilde{\omega}_{1,l}^2}. \tag{43}$$

With $\chi = 0$, we have that the ground state energies (43) stem from the interaction of the Klein-Gordon oscillator with a magnetic field and a Cornell-type scalar potential in the Minkowski space-time with a cosmic string, which is also a periodic function of the Aharonov-Bohm geometric quantum phase. Thus, the topology of the space-time also changes the pattern of oscillation of the ground state energies.

The corresponding ground state eigenfunctions using (41) are given by

$$\psi_{1,l} = x^{\sqrt{((l-\Phi-k\chi)^2/\alpha^2)+\eta_c^2}}\, e^{-(1/2)\left(x+\left(2m\eta_{1,L}/\tilde{\omega}_{1,l}^{3/2}\right)\right)x} \cdot \left[1 + \frac{1}{\sqrt{\tilde{\omega}_{1,l}}}\left\{\frac{2m\eta_c}{1+2j} + \frac{m\eta_{1,L}}{\tilde{\omega}_{1,l}}\right\}x\right], \tag{44}$$

where the potential parameter $\eta_L \longrightarrow \eta_{1,L}$is so adjusted that the first-order polynomial solution to the bound states can be obtained. Similarly, one can evaluate the energy levels and wave functions for $n = 2, 3$, and so on.

Now, we discuss below few cases of the above obtained energy eigenvalues.

Case 1. Linear scalar potential $S = \eta_L r$.

We consider here $\eta_c \longrightarrow 0$, that is, there is only a linear scalar potential into the relativistic quantum system.

Therefore, the energy eigenvalue Equation (40) becomes

$$E_{n,l}^2 = m^2 + k^2 + 2m\Omega + \frac{2m\omega_c}{\alpha}(l - k\chi - \Phi) + 2\tilde{\omega}\left(n + 1 + \frac{|l - k\chi - \Phi|}{\alpha}\right) - \frac{m^2\eta_L^2}{\tilde{\omega}^2}. \tag{45}$$

Equation (45) is the eigenvalue of a charged scalar particle in the presence of external fields including an internal magnetic flux field in a cosmic string space-time with a space-like dislocation subject to a linear scalar potential. For $\alpha \longrightarrow 1$, the energy eigenvalue Equation (45) reduces to the result obtained in Ref. [30] (see Equation (4) in Ref. [30]). With $\chi = 0$, we have that the energy eigenvalue (45) stems from the interaction of the Klein-Gordon oscillator with a magnetic field and a linear scalar potential in the Minkowski space-time with a cosmic string. Thus, the topology of the space-time also changes the pattern of oscillation of the ground state energies.

Case 2. Absence of external magnetic field, $B_0 = 0$.

We choose here $B_0 \longrightarrow 0$, that is, there are no external fields into the considered relativistic quantum system. In that case, the energy eigenvalue Equation (40) becomes

$$E_{n,l}^2 = m^2 + k^2 + 2\sqrt{m^2\Omega^2 + \eta_L^2} \cdot \left(n + 1 + \sqrt{\frac{(l - k\chi - \Phi)^2}{\alpha^2} + \eta_c^2}\right) - \frac{m^2\eta_L^2}{m^2\Omega^2 + \eta_L^2} + 2\eta_c\eta_L + 2m\Omega. \tag{46}$$

Equation (46) is the energy eigenvalue of the Klein-Gordon oscillator field in the presence of an internal magnetic flux field which is a cosmic string space-time with a space-like dislocation subject to a Cornell-type scalar potential. For $\alpha \longrightarrow 1$, the energy eigenvalue Equation (46) reduces to the result obtained in Ref. [31]. With $\chi = 0$, we have that the energy eigenvalue (46) stems from the interaction of the Klein-Gordon oscillator with a Cornell-type scalar potential in the Minkowski space-time with a cosmic string. Thus, the topology of the space-time also changes the pattern of oscillations of the ground state energies.

Case 3. Zero dislocation parameter $\chi = 0$.

We choose here the zero torsion parameter, $\chi \longrightarrow 0$, into the considered relativistic system. In that case, the energy eigenvalue Equation (40) becomes

$$E_{n,l}^2 = m^2 + k^2 + 2\eta_c\eta_L + 2m\Omega + \frac{2m\omega_c}{\alpha}(l - \Phi) + 2\tilde{\omega}\left(n + 1 + \sqrt{\frac{(l - \Phi)^2}{\alpha^2} + \eta_c^2}\right) - \frac{m^2\eta_L^2}{\tilde{\omega}^2}. \tag{47}$$

Equation (47) is the energy eigenvalue of a massive charged particle in the presence of an external uniform magnetic field including an internal magnetic flux subject to a Cornell-type scalar potential in a cosmic string space-time.

In all the above cases, we see that the relativistic energy eigenvalue depends on the geometric quantum phase [48] which gives rise to the analogous effect to the Aharonov-Bohm effect for bound states. Besides, we have that $E_{n,\bar{l}}(\Phi_B + \Phi_0) = E_{n,\bar{l}\mp\tau}(\Phi_B)$ where $\Phi_0 = \pm(2\pi\alpha/e)\tau$ with $\tau = 1, 2, 3, \cdots$ and $\bar{l} = l/\alpha$. It is observed in Cases 1 and 2 that the angular momentum eigenvalue l is shifted, $l \longrightarrow l_{\text{eff}} = (1/\alpha)(l - \Phi - k\chi)$, whereas in Case 3, it is $l \longrightarrow l' = (1/\alpha)(l - \Phi)$, an effective angular quantum number. As done earlier, one can evaluate the individual energy level and eigenfunction one by one.

3. Conclusions

We have investigated the effect of torsion and topological defects that stem from a space-time with a space-like dislocation on the interactions between an electrically charged particle and an external uniform magnetic field. Besides, we have assumed that the topological defects have an internal magnetic flux. By solving the Klein-Gordon oscillator equation analytically in Section 2.1, we have obtained the relativistic energy eigenvalue Equation (25) and the corresponding eigenfunction Equation (26). We have shown that for $\alpha \longrightarrow 1$, the energy eigenvalue reduces to the result obtained in Ref. [30]. We have seen in Equation (25) that there exists an effective angular momentum quantum number, $l \longrightarrow l_{\text{eff}} = (1/\alpha)(l - k\chi - (\phi_B/(2\pi/e)))$. Thus, the relativistic energy eigenvalue depends on the geometric quantum phase [48]. This dependence of the energy eigenvalue on the geometric quantum phase gives rise to the analogue effect to the Aharonov-Bohm effect for bound states [49–51]. Thus, we have that $E_{n,\bar{l}}(\Phi_B + \Phi_0) = E_{n,\bar{l}\mp\tau}(\Phi_B)$, where $\Phi_0 = \pm(2\pi\alpha/e)\tau$ with $\tau = 1, 2, 3, \cdots$ and $\bar{l} = l/\alpha$. For the zero torsion parameter $\chi \longrightarrow 0$, we have also obtained the energy eigenvalue Equation (27) which is the extended result in comparison with those obtained in [7] in a cosmic string space-time in the presence of external fields including an internal magnetic flux field. We have seen that the presence of torsion $\chi \neq 0$ in the space-time modifies the degeneracy of the relativistic energy levels. Besides, the presence of torsion in the space-time changes the pattern of oscillation of the energy levels.

We have extended our above discussion to investigate the behaviour of this relativistic system under the influence of a Cornell-type scalar potential in Section 2.2. We have solved the Klein-Gordon oscillator equation in the cosmic string space-time with a space-like dislocation and obtained the energy eigenvalue Equation (40). We have seen that for $\alpha \longrightarrow 1$ and $\eta_c \longrightarrow 0$, this energy eigenvalue reduces to the result obtained in Ref. [30]. Furthermore, in the absence of external fields ($B_0 \longrightarrow 0$), this energy eigenvalue Equation (40) reduces to the result obtained in Ref. [31]. Thus, we have observed that the relativistic energy eigenvalue Equation (40) is the extended results in comparison to those obtained in Refs. [30, 31]. Also, we have seen that the relativistic energy eigenvalue Equation (40) depends on the Aharonov-Bohm geometric quantum phase [48]. This dependence of the relativistic energy eigenvalue on the geometric quantum phase gives rise to the analogue effect to the Aharonov-Bohm effect for bound states [49–51]. We have that $E_{n,\bar{l}}(\Phi_B + \Phi_0) = E_{n,\bar{l}\mp\tau}(\Phi_B)$ where $\Phi_0 = \pm(2\pi\alpha/e)\tau$ with $\tau = 1, 2, 3, \cdots$ and $\bar{l} = l/\alpha$. Thus, we have seen that the presence of torsion ($\chi \neq 0$) in the space-time modifies the degeneracy of the relativistic energy levels. Besides, the presence of torsion in the space-time changes the pattern of oscillation of the energy levels. For $\chi = 0$, we have also obtained the relativistic energy eigenvalue Equation (47) of a massive charged particle in the presence of external fields including an internal magnetic flux field in a cosmic string space-time subject to a Cornell-type scalar potential. We have seen that the energy eigenvalue depends on the geometric quantum phase [48] which gives rise to the analogue effect to the Aharonov-Bohm effect for bound states [49–51].

In recent years, thermodynamic properties of quantum systems [86–90], quantum Hall effect [23, 91, 92], and displaced Fock states [93, 94] and the possibility of building a coherent state [95–98] have attracted a great current research interest in the literature. It is well known in nonrelativistic quantum mechanics that the Landau quantization is the simplest system that we can work with in the studies of the quantum Hall effect. Therefore, the relativistic systems analyzed in this work may be used for investigating the influence of torsion and topological defects (cosmic string) as well as the potential for searching the relativistic analogue to the quantum Hall effect, coherent states, and displaced Fock states in topological defects of space-time with a space-like dislocation. So the results given in this paper with those in Refs. [7, 30, 31] would present the above interesting effects.

Data Availability

No data have been used to prepare this manuscript.

Conflicts of Interest

The author declares that there is no conflict of interest regarding publication this paper.

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