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Research Article

Existence and Uniqueness of Solutions for Coupled Impulsive Fractional Pantograph Differential Equations with Antiperiodic Boundary Conditions

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In this paper, we investigate the solutions of coupled fractional pantograph differential equations with instantaneous impulses. The work improves some existing results and contributes toward the development of the fractional differential equation theory. We first provide some definitions that will be used throughout the paper; after that, we give the existence and uniqueness results that are based on Banach's contraction principle and Krasnoselskii's fixed point theorem. Two examples are given in the last part to support our study.

1. Introduction

Fractional differential equations (FDEs) involve fractional derivatives of the form d^α/dx^α, which are defined for $\alpha > 0$, where α is not necessarily an integer. They are generalizations of the ordinary differential equations to a random (noninteger) order. These FDEs have attracted considerable interest due to their ability to model complex phenomena. The fractional differential operators are global, and they are used to model several physical phenomena because they give accurate results. For the new readers that are interested in the fractional calculus theory in a more general concept, please see [1–6] and the references therein. Our work is concerned with impulsive coupled systems of pantograph FDEs. Impulsive FDEs have found applications in many areas such as business mathematics, management sciences, and population dynamics. Some physical problems have sudden changes and discontinuous jumps. To model these problems, we impose impulsive conditions on the differential equations at discontinuity points; for more details about impulsive fractional differential equations, we give the following references [7–13].

Many papers have studied impulsive fractional differential equations with antiperiodic boundary conditions, and results on the existence and uniqueness have been given (see [14–17]). For example, recently, Zuo et al. [18] investigated the existence results for an equation with impulsive and antiperiodic boundary conditions given by

$$\begin{cases} {}^cD^\alpha x(t) + \gamma x(t) = f(t, x(t), Ax(t), Bx(t)), & t \in J = [0,1],\ t \neq t_i,\ i = 1, 2, \cdots, m, \\ \Delta x_{|t=t_i}(0) = I_i(x(t_i)), & i = 1, \cdots, m, \\ x(0) = -x(1), \end{cases} \tag{1}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order $\alpha \in (0, 1)$, $\gamma > 0$, $I_i \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_{i+1} = 1$, the function f is in $C(J \times \mathbb{R}^3, \mathbb{R})$, $\Delta x_{|t=t_i}$ denotes the jump of $x(t)$ at $t = t_i$, and A and B are linear operators. The authors established the existence and uniqueness under some conditions using Banach's and Krasnoselskii's fixed point theorems.

On the other hand, in the deterministic situation, there is a very special case of delay differential equations known as the pantograph equations given by

$$\begin{cases} g'(t) = kg(t) + lg(\lambda t), & t \in [0, b],\ b > 0,\ 0 < \lambda < 1, \\ g(0) = g_0. \end{cases} \tag{2}$$

These equations are also called equations with proportional delays. Pantographs are special devices mounted on electric trains to collect current from one or several contact wires. They consist of a pantograph head, frame, base, and drive system, and their geometrical shape is variable. But it is recently being used in electric trains. Many researchers have investigated the pantograph differential equations and their properties; see [19–21].

Motivated by all the previous works, we consider in this paper coupled impulsive fractional pantograph differential equations with antiperiodic boundary conditions as follows:

$$\begin{cases} {}^cD^{\alpha_1}x(t)+\gamma_1 x(t)=f_1(t,x(t),x(\lambda_1 t),y(t)), & t\in J,\ t\neq t_i,\ i=1,2,\cdots,m,\ 0<\alpha_1<1,\ 0<\lambda_1<1,\\ {}^cD^{\alpha_2}y(t)+\gamma_2 y(t)=f_2(t,x(t),y(t),y(\lambda_2 t)), & t\in J,\ t\neq t_j,\ j=1,2,\cdots,n,\ 0<\alpha_2<1,\ 0<\lambda_2<1,\\ \Delta x_{|t=t_i}(0)=I_i(x(t_i)), & i=1,\cdots,m,\\ \Delta y_{|t=t_j}(0)=I_j\left(x\left(t_j\right)\right), & j=1,\cdots,n,\\ a_1x(0)+b_1x(1)=0, & a_1\geq b_1>0,\\ a_2y(0)+b_2y(1)=0, & a_2\geq b_2>0, \end{cases} \tag{3}$$

where ${}^cD^{\alpha_1}$ and ${}^cD^{\alpha_2}$ are the Caputo fractional derivatives of orders α_1 and α_2, respectively, $f_1, f_2 : J\times\mathbb{R}^3\to\mathbb{R}$ are two continuous functions, $J=[0,1]$, $\Delta x_{|t=t_i}=x(t_i^+)-x(t_i^-)$, with $x(t_i^+)$ and $x(t_i^-)$ representing the right and left limits of $x(t)$ at $t=t_i$, $i=1,\cdots,m$, and also $\Delta y_{|t=t_j}=y(t_j^+)-y(t_j^-)$, with $y(t_j^+)$ and $y(t_j^-)$ representing the right and left limits of $y(t)$ at $t=t_j$, $j=1,\cdots,n$.

The objective of this paper is to establish the existence and uniqueness results of the solutions of problem (3) by means of Banach's contraction principle and Krasnoselskii's fixed point theorem.

The main contributions of this paper are as follows:

(i) We consider a new system of impulsive pantograph fractional differential equations

(ii) We consider antiperiodic boundary value conditions with a more general form

This paper contributes toward the development of qualitative analysis of impulsive fractional differential equations.

This paper is organized as follows: in Section 2, we give some definitions and useful lemmas that will be used throughout the work; after that, in Section 3, we will establish the existence and uniqueness results by means of the fixed point theorems; last but not least, in Section 4, we give two illustrative examples.

2. Preliminaries and Lemmas

Let $J_0=(0,t_1]$, $J_1=(t_1,t_2]$, $\cdots$, $J_m=(t_m,1]$, and $PC(J,X)=\{x:J\to\mathbb{R}:x\in C(J_i,\mathbb{R})\}$, where $i=0,1,2,\cdots,m$, $x(t_i^+)$ and $x(t_i^-)$ exist, $i=1,\cdots,m$, is a space of continuous real-valued functions on the interval J, and $x(t_i^-)=x(t_i)$.

Similarly, we define $PC(J,Y)=\{y:J\to\mathbb{R}:y\in C(J_j,\mathbb{R})\}$, where $j=0,1,2,\cdots,n$, $y(t_j^+)$ and $y(t_j^-)$ exist, $j=1,\cdots,n$, is a space of continuous real-valued functions on the interval J, and $y(t_j^-)=y(t_j)$.

Then, clearly, $PC(J,X)$ and $PC(J,Y)$ are two Banach spaces with the norms $\|x\|=\sup_{t\in[0,1]}|x(t)|$ and $\|y\|=\sup_{t\in[0,1]}|y(t)|$, respectively.

Consequently, the space $PC(J,X)\times PC(J,Y)$ is a Banach space with the norm $\|(x,y)\|=\|x\|+\|y\|$.

We note that the space $L^p(J,\mathbb{R})$ is a Banach space of Lebesque measurable functions with $\|.\|_{L^p(J)}<\infty$.

Definition 1 (see [1]). The fractional integral of order α with the lower limit zero for a function f is defined as

$$I_{0^+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds, \quad \alpha>0,\ n\in\mathbb{N}, \tag{4}$$

provided the right-hand side is pointwise defined on $[0,\infty)$, where $\Gamma(.)$ denotes the Gamma function.

Definition 2 (see [1]). The Riemann-Liouville derivative of order α with the lower limit zero for a function f is defined as

$${}^LD_{0^+}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}f(s)ds, \quad \alpha>0,\ n-1<\alpha<n, \tag{5}$$

provided the function f is absolutely continuous up to order $(n-1)$ derivatives, where $\Gamma(.)$ denotes the Gamma function.

Definition 3 (see [1]). The Caputo derivative of order $\alpha>0$ with the lower limit zero for a function f is defined as

$${}^cD_{0^+}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}\left(f(s)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right)ds, \quad n=[\alpha]+1,\ n\in\mathbb{N}, \tag{6}$$

provided the function $f : [0,\infty) \to R$, where $\Gamma(.)$ denotes the Gamma function.

Definition 4. A couple (x, y) is a solution of problem (3) if it satisfies the equations

$$\begin{cases} {}^cD^{\alpha_1}x(t) + \gamma_1 x(t) = f_1(t, x(t), x(\lambda_1 t), y(t)), & t \neq t_i,\ i = 1, 2, \cdots, m, \\ {}^cD^{\alpha_2}y(t) + \gamma_2 y(t) = f_2(t, x(t), y(t), y(\lambda_2 t)), & t \neq t_j,\ j = 1, 2, \cdots, n, \end{cases} \tag{7}$$

a.e. on J, and the conditions $\Delta x_{|t=t_i}(0) = I_i(x(t_i))$, $\Delta y_{|t=t_j}(0) = I_j(y(t_j))$, $i = 1, \cdots, m$, $j = 1, \cdots, n$ and $a_1 x(0) + b_1 x(1) = 0$, $a_2 y(0) + b_2 y(1) = 0$, $a_1 \geq b_1 > 0$, $a_2 \geq b_2 > 0$.

Lemma 5 (see [22]). *The nonnegative functions E_α and $E_{\alpha,\alpha}$ given by*

$$\begin{aligned} E_\alpha(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \\ E_{\alpha,\alpha}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \alpha)}, \end{aligned} \tag{8}$$

have the following properties:

(1) For any $\gamma > 0$ and $t \in J$,

$$\begin{aligned} E_\alpha(-t^\alpha \gamma) &\leq 1, \\ E_{\alpha,\alpha}(-t^\alpha \gamma) &\leq \frac{1}{\Gamma(\alpha)}. \end{aligned} \tag{9}$$

In addition, we have $E_\alpha(0) = 1$ and $E_{\alpha,\alpha}(0) = 1/\Gamma(\alpha)$.

(2) For any $\gamma > 0$ and $t_1, t_2 \in J$,

$$\begin{aligned} E_\alpha(-t_2^\alpha \gamma) &\to E_\alpha(-t_1^\alpha \gamma), \quad \text{as } t_2 \to t_1, \\ E_{\alpha,\alpha}(-t_2^\alpha \gamma) &\to E_{\alpha,\alpha}(-t_1^\alpha \gamma), \quad \text{as } t_2 \to t_1. \end{aligned} \tag{10}$$

(3) For any $\gamma > 0$ and $t_1, t_2 \in J$ such that $t_1 \leq t_2$,

$$\begin{aligned} E_\alpha(-t_2^\alpha \gamma) &\leq E_\alpha(-t_1^\alpha \gamma), \\ E_{\alpha,\alpha}(-t_2^\alpha \gamma) &\leq E_{\alpha,\alpha}(-t_1^\alpha \gamma). \end{aligned} \tag{11}$$

Lemma 6 (see [23]). *Let M be a closed, convex, and nonempty subset of a Banach space X, and let F_1 and F_2 be operators such that*

(1) $F_1 x + F_2 y \in M$ whenever $x, y \in M$

(2) F_1 is compact and continuous

(3) F_2 is a contraction mapping

Then, there exists $z \in M$ such that $z = F_1 z + F_2 z$.

Lemma 7 (see [24]). *Let X be a Banach space, and let $J = [0, T]$. Suppose that $W \subset PC(J, X)$ satisfies the following conditions:*

(1) W is a uniformly bounded subset of $PC(J, X)$

(2) W is equicontinuous in (t_i, t_{i+1}), $i = 0, 1, \cdots, m$, where $t_0 = 0$ and $t_{m+1} = T$

(3) Its t-sections $W(t) = \{x(t): x \in W, t/\{t_1, \cdots, t_m\}$, $W(t_i^+) = \{x(t_i^+): x \in W\}$, and $W(t_i^-) = \{x(t_i^-): x \in W\}$ are relatively compact subsets of X

Then, W is a relatively compact subset of $PC(J, X)$.

Lemma 8 (see [25]). *Let $f_1, f_2 \to \mathbb{R}$ be two continuous functions. The couple (x, y) given by*

$$x(t) = \begin{cases} \dfrac{-E_{\alpha_1}(-\gamma_1) E_{\alpha_1}(-t^{\alpha_1}\gamma_1)}{1 + E_{\alpha_1}(-\gamma_1)} \displaystyle\sum_{k=1}^{m} \frac{u_k}{E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)} + \int_0^t (t-s)^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(-(t-s)^{\alpha_1}\gamma_1) f_1(s) ds - \frac{E_{\alpha_1}(-t^{\alpha_1}\gamma_1)}{1 + E_{\alpha_1}(-\gamma_1)} \int_0^1 (1-s)^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(-(1-s)^{\alpha_1}\gamma_1) f_1(s) ds, & t \in J_0, \\ \dfrac{E_{\alpha_1}(-t^{\alpha_1}\gamma_1)}{1 + E_{\alpha_1}(-\gamma_1)} \left(\displaystyle\sum_{k=1}^{m} \frac{u_k}{E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)} - \int_0^1 (1-s)^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(-(1-s)^{\alpha_1}\gamma_1) f_1(s) ds \right) - E_{\alpha_1}(-t^{\alpha_1}\gamma_1) \displaystyle\sum_{p=i+1}^{m} \frac{u_p}{E_{\alpha_1}(-t_p^{\alpha_1}\gamma_1)} + \int_0^t (t-s)^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(-(t-s)^{\alpha_1}\gamma_1) f_1(s) ds, & t \in J_i,\ i = 1, 2, \cdots, m-1, \\ \dfrac{E_{\alpha_1}(-t^{\alpha_1}\gamma_1)}{1 + E_{\alpha_1}(-\gamma_1)} \left(\displaystyle\sum_{k=1}^{m} \frac{u_k}{E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)} - \int_0^1 (1-s)^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(-(1-s)^{\alpha_1}\gamma_1) f_1(s) ds \right) + \int_0^t (t-s)^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(-(t-s)^{\alpha_1}\gamma_1) f_1(s) ds, & t \in J_m, \end{cases} \tag{12}$$

$$y(t) = \begin{cases} \dfrac{-E_{\alpha_2}(-\gamma_2) E_{\alpha_2}(-t^{\alpha_2}\gamma_2)}{1 + E_{\alpha_2}(-\gamma_2)} \displaystyle\sum_{k=1}^{n} \frac{v_k}{E_{\alpha_2}(-t_k^{\alpha_2}\gamma_2)} + \int_0^t (t-s)^{\alpha_2 - 1} E_{\alpha_2,\alpha_2}(-(t-s)^{\alpha_2}\gamma_2) f_2(s) ds - \frac{E_{\alpha_2}(-t^{\alpha_2}\gamma_2)}{1 + E_{\alpha_2}(-\gamma_2)} \int_0^1 (1-s)^{\alpha_2 - 1} E_{\alpha_2,\alpha_2}(-(1-s)^{\alpha_2}\gamma_2) f_2(s) ds, & t \in J_0, \\ \dfrac{E_{\alpha_2}(-t^{\alpha_2}\gamma_2)}{1 + E_{\alpha_2}(-\gamma_2)} \left(\displaystyle\sum_{k=1}^{n} \frac{v_k}{E_{\alpha_2}(-t_k^{\alpha_2}\gamma_2)} - \int_0^1 (1-s)^{\alpha_2 - 1} E_{\alpha_2,\alpha_2}(-(1-s)^{\alpha_2}\gamma_2) f_2(s) ds \right) - E_{\alpha_2}(-t^{\alpha_2}\gamma_2) \displaystyle\sum_{p=j+1}^{m} \frac{v_p}{E_{\alpha_2}(-t_p^{\alpha_2}\gamma_2)} + \int_0^t (t-s)^{\alpha_2 - 1} E_{\alpha_2,\alpha_2}(-(t-s)^{\alpha_2}\gamma_2) f_2(s) ds, & t \in J_j,\ j = 1, 2, \cdots, n-1, \\ \dfrac{E_{\alpha_2}(-t^{\alpha_2}\gamma_2)}{1 + E_{\alpha_2}(-\gamma_2)} \left(\displaystyle\sum_{k=1}^{n} \frac{v_k}{E_{\alpha_2}(-t_k^{\alpha_2}\gamma_2)} - \int_0^1 (1-s)^{\alpha_2 - 1} E_{\alpha_2,\alpha_2}(-(1-s)^{\alpha_2}\gamma_2) f_2(s) ds \right) + \int_0^t (t-s)^{\alpha_2 - 1} E_{\alpha_2,\alpha_2}(-(t-s)^{\alpha_2}\gamma_2) f_s(s) ds, & t \in J_n, \end{cases} \tag{13}$$

is a solution of the impulsive problem

$$\begin{cases} {}^cD^{\alpha_1}x(t) + \gamma_1 x(t) = f_1(t), & t \in J,\ t \neq t_i,\ i = 1, 2, \cdots, m, \\ {}^cD^{\alpha_2}y(t) + \gamma_2 y(t) = f_2(t), & t \in J,\ t \neq t_j,\ j = 1, 2, \cdots, n, \\ \Delta x_{|t=t_i}(0) = u_i, & i = 1, \cdots, m, \\ \Delta y_{|t=t_j}(0) = v_j, & j = 1, \cdots, n, \\ x(0) + x(1) = 0, \\ y(0) + y(1) = 0. \end{cases} \tag{14}$$

It follows from Lemma 8, and by using the boundary conditions $a_1x(0) + b_1x(1) = 0$ and $a_2y(0) + b_2y(1) = 0$, that the solution of (3) can be expressed as follows:

$$x(t) = \begin{cases} \dfrac{-E_{\alpha_1}(-\gamma_1)E_{\alpha_1}(-t^{\alpha_1}\gamma_1)}{1 + E_{\alpha_1}(-\gamma_1)} \displaystyle\sum_{k=1}^{m} \frac{I_k(x(t_k))}{E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)} + \int_0^t (t-s)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(-(t-s)^{\alpha_1}\gamma_1) f_1(s, x(s), x(\lambda_1 s), y(s)) ds - \frac{E_{\alpha_1}(-t^{\alpha_1}\gamma_1)}{1 + \sigma_1 E_{\alpha_1}(-\gamma_1)} \sigma_1 \int_0^1 (1-s)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(-(1-s)^{\alpha_1}\gamma_1) f_1(s, x(s), x(\lambda_1 s), y(s)) ds, & t \in J_0, \\ \dfrac{E_{\alpha_1}(-t^{\alpha_1}\gamma_1)}{1 + \sigma_1 E_{\alpha_1}(-\gamma_1)} \left(\displaystyle\sum_{k=1}^{m} \frac{I_k(x(t_k))}{E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)} - \sigma_1 \int_0^1 (1-s)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(-(1-s)^{\alpha_1}\gamma_1) f_1(s, x(s), x(\lambda_1 s), y(s)) ds \right) - E_{a_1}(-t^{\alpha_1}\gamma_1) \displaystyle\sum_{p=i+1}^{m} \frac{I_p(x(t_p))}{E_{\alpha_1}(-t_p^{\alpha_1}\gamma_1)} + \int_0^t (t-s)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(-(t-s)^{\alpha_1}\gamma_1) f_1(s, x(s), x(\lambda_1 s), y(s)) ds, & t \in J_i,\ i = 1, 2, \cdots, m-1, \\ \dfrac{E_{\alpha_1}(-t^{\alpha_1}\gamma_1)}{1 + \sigma_1 E_{\alpha_1}(-\gamma_1)} \left(\displaystyle\sum_{k=1}^{m} \frac{I_k(x(t_k))}{E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)} - \sigma_1 \int_0^1 (1-s)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(-(1-s)^{\alpha_1}\gamma_1) f_1(s, x(s), x(\lambda_1 s), y(s)) ds \right) + \displaystyle\int_0^t (t-s)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(-(t-s)^{\alpha_1}\gamma_1) f_1(s, x(s), x(\lambda_1 s), y(s)) ds, & t \in J_m, \end{cases} \tag{15}$$

$$y(t) = \begin{cases} \dfrac{-E_{\alpha_2}(-\gamma_2)E_{\alpha_2}(-t^{\alpha_2}\gamma_2)}{1 + E_{\alpha_2}(-\gamma_2)} \displaystyle\sum_{k=1}^{n} \frac{I_k(y(t_k))}{E_{\alpha_2}(-t_k^{\alpha_2}\gamma_2)} + \int_0^t (t-s)^{\alpha_2-1} E_{\alpha_2,\alpha_2}(-(t-s)^{\alpha_2}\gamma_2) f_2(s, x(s), y(s), y(\lambda_2 s)) ds - \frac{E_{\alpha_2}(-t^{\alpha_2}\gamma_2)}{1 + \sigma_2 E_{\alpha_2}(-\gamma_2)} \sigma_2 \int_0^1 (1-s)^{\alpha_2-1} E_{\alpha_2,\alpha_2}(-(1-s)^{\alpha_2}\gamma_2) f_2(s, x(s), y(s), y(\lambda_2 s)) ds, & t \in J_0, \\ \dfrac{E_{\alpha_2}(-t^{\alpha_2}\gamma_2)}{1 + \sigma_2 E_{\alpha_2}(-\gamma_2)} \left(\displaystyle\sum_{k=1}^{n} \frac{I_k(y(t_k))}{E_{\alpha_2}(-t_k^{\alpha_2}\gamma_2)} - \sigma_2 \int_0^1 (1-s)^{\alpha_1-1} E_{\alpha_2,\alpha_2}(-(1-s)^{\alpha_2}\gamma_2) f_2(s, x(s), y(s), y(\lambda_2 s)) ds \right) - E_{\alpha_2}(-t^{\alpha_2}\gamma_2) \displaystyle\sum_{p=j+1}^{n} \frac{I_p(y(t_p))}{E_{\alpha_2}(-t_p^{\alpha_2}\gamma_2)} + \int_0^t (t-s)^{\alpha_2-1} E_{\alpha_2,\alpha_2}(-(t-s)^{\alpha_2}\gamma_2) f_2(s, x(s), y(s), y(\lambda_2 s)) ds, & t \in J_j,\ j = 1, 2, \cdots, n-1, \\ \dfrac{E_{\alpha_2}(-t^{\alpha_2}\gamma_2)}{1 + \sigma_2 E_{\alpha_2}(-\gamma_2)} \left(\displaystyle\sum_{k=1}^{n} \frac{I_k(y(t_k))}{E_{\alpha_2}(-t_k^{\alpha_2}\gamma_2)} - \sigma_2 \int_0^1 (1-s)^{\alpha_2-1} E_{\alpha_2,\alpha_2}(-(1-s)^{\alpha_2}\gamma_2) f_2(s, x(s), y(s), y(\lambda_2 s)) ds \right) + \displaystyle\int_0^t (t-s)^{\alpha_2-1} E_{\alpha_2,\alpha_2}(-(t-s)^{\alpha_2}\gamma_2) f_2(s, x(s), y(s), y(\lambda_2 s)) ds, & t \in J_n, \end{cases} \tag{16}$$

where $\sigma_1 = b_1/a_1$ and $\sigma_2 = b_2/a_2$.

3. Main Results

Theorem 9. *We consider the following hypotheses:*

(H_1). The functions $f_1, f_2 : J \times \mathbb{R}^3 \to \mathbb{R}$ are continuous, and there exist two constants $L_1, L_2 > 0$ such that

$$\begin{aligned} |f_1(t, x, y, z) - f_1(t, u, v, w)| &\leq L_1(|x-u| + |y-v| + |z-w|), \\ |f_2(t, x, y, z) - f_2(t, u, v, w)| &\leq L_2(|x-u| + |y-v| + |z-w|), \end{aligned} \tag{17}$$

for all $t \in J$, $x, y, z, u, v, w \in \mathbb{R}$.

(H_2). $|I_i(x) - I_i(y)| \leq C_1|x-y|$ and

$$\left| I_j(x) - I_j(y) \right| \leq C_2|x-y|, \tag{18}$$

for all $x, y \in \mathbb{R}$, $i = 1, 2, \cdots, m$, and $j = 1, 2, \cdots, n$.

(H_3). We suppose that $(\mu_1 + \mu_2) < 1$.

Then, problem (3) has a unique solution (x^*, y^*).

Remark 10. The expressions of μ_1 and μ_2 are given in the proof.

Proof. We define the operator $T : PC(J, X) \times PC(J, Y) \to PC(J, X) \times PC(J, Y)$ by

$$T(x, y)(t) = (U(x, y)(t), V(x, y)(t)), \tag{19}$$

where

$$\begin{aligned} U(x, y)(t) = {} & \frac{E_{\alpha_1}(-t^{\alpha_1}\gamma_1)}{1 + \sigma_1 E_{\alpha_1}(-\gamma_1)} \left(\sum_{k=1}^{m} \frac{I_k(x(t_k))}{E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)} \right. \\ & - \sigma_1 \int_0^1 (1-s)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(-(1-s)^{\alpha_1}\gamma_1) f_1(s, x(s), \\ & \left. x(\lambda_1 s), y(s)) ds \right) - E_{\alpha_1}(-t^{\alpha_1}\gamma_1) \sum_{p=i+1}^{m} \frac{I_p(x(t_p))}{E_{\alpha_1}(-t_p^{\alpha_1}\gamma_1)} \\ & + \int_0^t (t-s)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(-(t-s)^{\alpha_1}\gamma_1) f_1(s, x(s), \\ & x(\lambda_1 s), y(s)) ds, \quad t \in [t_i, t_{i+1}),\ i = 0, 1, 2, \cdots, m, \end{aligned}$$

$$\begin{aligned} V(x, y)(t) = {} & \frac{E_{\alpha_2}(-t^{\alpha_2}\gamma_2)}{1 + \sigma_2 E_{\alpha_2}(-\gamma_2)} \left(\sum_{k=1}^{n} \frac{I_k(y(t_k))}{E_{\alpha_2}(-t_k^{\alpha_2}\gamma_2)} \right. \\ & - \sigma_2 \int_0^1 (1-s)^{\alpha_2-1} E_{\alpha_2,\alpha_2}(-(1-s)^{\alpha_2}\gamma_2) f_2(s, x(s), \end{aligned}$$

$$y(s), y(\lambda_2 s))ds\Bigg) - E_{\alpha_2}(-t^{\alpha_2}\gamma_2)\sum_{p=j+1}^{n}\frac{I_p(y(t_p))}{E_{\alpha_2}(-t_p^{\alpha_2}\gamma_2)} + \int_0^t (t-s)^{\alpha_2-1}E_{\alpha_2,\alpha_2}(-(t-s)^{\alpha_2}\gamma_2)f_2(s, x(s), y(s), y(\lambda_2 s))ds, \quad t\in[t_j, t_{j+1}),\ j=0,1,2,\cdots,n. \tag{20}$$

We show now that the operator T has a fixed point, which is a solution of problem (3).

Let $K_1 = \sup_{t\in J}|f_1(t,0,0,0)|$, $K_2 = \sup_{t\in J}|f_2(t,0,0,0)|$, $K_1^* = \max\{|I_i(0)|: i=1,2,\cdots,m\}$, and $K_2^* = \max\{|I_j(0)|: j=1,2,\cdots,n\}$.

We choose

$$r \ge \frac{\sum_{k=1}^m \left(K_1^*/\left|E_{\alpha_1}\left(-t_k^{\alpha_1}\gamma_1\right)\right|\right) + (K_1/\Gamma(\alpha_1+1))}{\left(\left|1+\sigma_1 E_{\alpha_1}(-\gamma_1)\right|/(4+2\sigma_1)\right) - \left(\sum_{k=1}^m \left(C_1/\left|E_{\alpha_1}\left(-t_k^{\alpha_1}\gamma_1\right)\right|\right) + (6L_1(\sigma_1+1)/(2+\sigma_1)\Gamma(\alpha_1+1))\right)} + \frac{\sum_{k=1}^n \left(K_2^*/\left|E_{\alpha_2}\left(-t_k^{\alpha_2}\gamma_2\right)\right|\right) + (K_2/\Gamma(\alpha_2+1))}{\left(\left|\left(1+\sigma_2 E_{\alpha_2}(-\gamma_2)\right)\right|/(4+2\sigma_2)\right) - \left(\sum_{k=1}^n \left(C_2/\left|E_{\alpha_2}\left(-t_k^{\alpha_2}\gamma_2\right)\right|\right) + (6L_2(\sigma_2+1)/(2+\sigma_2)\Gamma(\alpha_2+1))\right)}. \tag{21}$$

Firstly, we show that $TB_r \subset B_r$, where $B_r = \{(x,y)\in PC(J,X)\times PC(J,Y): \|(x,y)\| \le r\}$. It follows from the hypotheses above and Lemma 5 that for any $(x,y)\in B_r$, we have

$$\begin{aligned}
|U(x,y)(t)| &\le \left|E_{\alpha_1}(-t^{\alpha_1}\gamma_1)\right|\left[\left|\frac{1}{1+\sigma_1 E_{\alpha_1}(-\gamma_1)}\left(\sum_{k=1}^m \frac{I_k(x(t_k))}{E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)} - \sigma_1\int_0^1 (1-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(-(1-s)^{\alpha_1}\gamma_1)\cdot f_1(s,x(s),x(\lambda_1 s),y(s))ds\right) - \sum_{p=i+1}^m \frac{I_p(x(t_p))}{E_{\alpha_1}(-t_p^{\alpha_1}\gamma_1)}\right|\right] \\
&\quad + \left|\int_0^t (t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(-(t-s)^{\alpha_1}\gamma_1)f_1(s,x(s),x(\lambda_1 s),y(s))ds\right| \\
&\le \frac{1}{\left|1+\sigma_1 E_{\alpha_1}(-\gamma_1)\right|}\left(\sum_{k=1}^m \frac{|I_k(x(t_k))|}{\left|E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)\right|} + \frac{\sigma_1}{\Gamma(\alpha_1)}\cdot\int_0^1 (1-s)^{\alpha_1-1}|f_1(s,x(s),x(\lambda_1 s),y(s))|ds\right) \\
&\quad + \sum_{k=1}^m \frac{|I_k(x(t_k))|}{\left|E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)\right|} + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\cdot|f_1(s,x(s),x(\lambda_1 s),y(s))|ds \le \frac{1+\left|1+\sigma_1 E_{\alpha_1}(-\gamma_1)\right|}{\left|1+\sigma_1 E_{\alpha_1}(-\gamma_1)\right|} \\
&\quad \cdot\left(\sum_{k=1}^m \frac{|I_k(x(t_k)) - I_k(0)| + K_1^*}{\left|E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)\right|}\right) + \frac{\sigma_1}{\Gamma(\alpha_1)\left|1+\sigma_1 E_{\alpha_1}(-\gamma_1)\right|} \\
&\quad \times\int_0^1 (1-s)^{\alpha_1-1}[|f_1(s,x(s),x(\lambda_1 s),y(s)) - f(s,0,0,0)| + |f(s,0,0,0)|]ds + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}[|f_1(s,x(s),x(\lambda_1 s),y(s)) - f(s,0,0,0)| + |f(s,0,0,0)|]ds \\
&\le \frac{(2+\sigma_1)}{\left|1+\sigma_1 E_{\alpha_1}(-\gamma_1)\right|}\sum_{k=1}^m \frac{C_1 r + K_1^*}{\left|E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)\right|} + \frac{K_1}{\Gamma(\alpha_1+1)\left|1+\sigma_1 E_{\alpha_1}(-\gamma_1)\right|} + \frac{K_1}{\Gamma(\alpha_1+1)} \\
&\quad + \frac{\sigma_1}{\Gamma(\alpha_1)\left|1+\sigma_1 E_{\alpha_1}(-\gamma_1)\right|}\int_0^1 (1-s)^{\alpha_1-1}L_1(|x(s)| + |x(\lambda_1 s)| + |y(s)|)ds + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}L_1(|x(s)| + |x(\lambda_1 s)| + |y(s)|)ds \\
&\le \frac{(2+\sigma_1)}{\left|1+\sigma_1 E_{\alpha_1}(-\gamma_1)\right|}\left[\sum_{k=1}^m \frac{C_1 r + K_1^*}{\left|E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)\right|} + \frac{K_1}{\Gamma(\alpha_1+1)}\right] + \frac{3rL_1\sigma_1}{\Gamma(\alpha_1+1)\left|1+\sigma_1 E_{\alpha_1}(-\gamma_1)\right|} + \frac{3rL_1}{\Gamma(\alpha_1+1)} \le \frac{(2+\sigma_1)}{\left|1+\sigma_1 E_{\alpha_1}(-\gamma_1)\right|} \\
&\quad \cdot\left[\sum_{k=1}^m \frac{K_1^*}{\left|E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)\right|} + \frac{K_1}{\Gamma(\alpha_1+1)} + \left(\sum_{k=1}^m \frac{C_1}{\left|E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)\right|} + \frac{6L_1(\sigma_1+1)}{(2+\sigma_1)\Gamma(\alpha_1+1)}\right)r\right] \le \frac{r}{2}.
\end{aligned} \tag{22}$$

Similarly, we show that

$$|V(x,y)(t)| \le \frac{(2+\sigma_2)}{\left|1+\sigma_2 E_{\alpha_2}(-\gamma_2)\right|}\left[\sum_{k=1}^n \frac{K_2^*}{\left|E_{\alpha_2}(-t_k^{\alpha_2}\gamma_2)\right|} + \frac{K_2}{\Gamma(\alpha_1+1)} + \left(\sum_{k=1}^n \frac{C_2}{\left|E_{\alpha_2}(-t_k^{\alpha_2}\gamma_2)\right|} + \frac{6L_2(\sigma_2+1)}{(2+\sigma_2)\Gamma(\alpha_2+1)}\right)r\right] \le \frac{r}{2}. \tag{23}$$

Finally,

$$|T(x,y)(t)| \le |U(x,y)(t)| + |V(x,y)(t)| \le r, \tag{24}$$

which implies that $TB_r \subset B_r$.

Next, we show that the operator T is a contraction; we let $(x,y), (\bar{x},\bar{y}) \in X\times Y$; then, for $t\in J$, we have

$$|U(x,y)(t) - U(\bar{x},\bar{y})(t)| = \left|\frac{E_{\alpha_1}(-t^{\alpha_1}\gamma_1)}{1+\sigma_1 E_{\alpha_1}(-\gamma_1)}\left(\sum_{k=1}^m \frac{I_k(x(t_k)) - I_k(\bar{x}(t_k))}{E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)}\right.\right.$$

$$-\sigma_1\int_0^1 (1-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(-(1-s)^{\alpha_1}\gamma_1)(f_1(s,x(s),x(\lambda_1 s),y(s)) - f_1(s,\bar{x}(s),\bar{x}(\lambda_1 s),\bar{y}(s)))ds\Bigg) - E_{\alpha_1}(-t^{\alpha_1}\gamma_1)\sum_{p=i+1}^{m} \cdot \frac{I_p(x(t_p)) - I_p(\bar{x}(t_p))}{E_{\alpha_1}(-t_p^{\alpha_1}\gamma_1)} + \int_0^t (t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1} \cdot (-(t-s)^{\alpha_1}\gamma_1)(f_1(s,x(s),x(\lambda_1 s),y(s)) - f_1(s,\bar{x}(s),\bar{x}(\lambda_1 s),\bar{y}(s)))ds| \le \left(\frac{1}{|1+\sigma_1 E_{\alpha_1}(-\gamma_1)|}+1\right) \cdot \sum_{k=1}^{m}\frac{C_1|x(t_k)) - \bar{x}(t_k)|}{E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)} + \frac{\sigma_1}{\Gamma(\alpha_1)\left|1+\sigma_1 E_{\alpha_1(-\gamma_1)}\right|} \cdot \int_0^1 (1-s)^{\alpha_1-1}L_1(|x(s)-\bar{x}(s)| + |x(\lambda_1 s) - \bar{x}(\lambda_1 s)| + |y(s)-\bar{y}(s)|)ds) + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}L_1 \cdot (|x(s)-\bar{x}(s)| + |x(\lambda_1 s)-\bar{x}(\lambda_1 s)| + |y(s)-\bar{y}(s)|)ds) \le \frac{(2+\sigma_1)}{|1+\sigma_1 E_{\alpha_1}(-\gamma_1)|}\sum_{k=1}^{m}\frac{C_1}{E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)}\|x-\bar{x}\| + \frac{3L_1\sigma_1}{\Gamma(\alpha_1+1)|1+\sigma_1 E_{\alpha_1}(-\gamma_1)|}(\|x-\bar{x}\| + \|y-\bar{y}\|) + \frac{3L_1}{\Gamma(\alpha_1+1)}(\|x-\bar{x}\| + \|y-\bar{y}\|) \le \frac{(2+\sigma_1)}{|1+\sigma_1 E_{\alpha_1}(-\gamma_1)|} \cdot \left(\sum_{k=1}^{m}\frac{C_1}{E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)} + \frac{6L_1(\sigma_1+1)}{(2+\sigma_1)\Gamma(\alpha_1+1)}\right) \cdot (\|x-\bar{x}\| + \|y-\bar{y}\|) = \mu_1(\|x-\bar{x}\| + \|y-\bar{y}\|). \tag{25}$$

With a similar method, we also get

$$|V(x,y)(t) - V(\bar{x},\bar{y})(t)| \le \frac{(2+\sigma_2)}{|1+\sigma_2 E_{\alpha_2}(-\gamma_2)|}\left(\sum_{k=1}^{n}\frac{C_2}{E_{\alpha_2}(-t_k^{\alpha_2}\gamma_2)} + \frac{6L_2(\sigma_2+1)}{(2+\sigma_2)\Gamma(\alpha_2+1)}\right) \cdot (\|x-\bar{x}\| + \|y-\bar{y}\|) = \mu_2(\|x-\bar{x}\| + \|y-\bar{y}\|). \tag{26}$$

Finally, we can obtain

$$|T(x,y)(t) - T(\bar{x},\bar{y})(t)| \le (\mu_1+\mu_2)(\|x-\bar{x}\| + \|y-\bar{y}\|). \tag{27}$$

And since $(\mu_1+\mu_2) < 1$, then the operator T is a contraction.

Therefore, we conclude by Banach's contraction mapping principle that T has a fixed point which is the unique solution (x^*, y^*) of problem (3). The proof is now completed.

Next, we present a result based on Krasnoselskii's fixed point theorem.

Theorem 11. *Assume that the condition (H_2) and the following additional conditions are satisfied:*

(H_4). Two functions $\varphi_1, \psi_1 \in L^{1/\rho_1}(0,+\infty)(0<\rho_1<\alpha_1<1)$, $\varphi_2, \psi_2 \in L^{1/\rho_2}(0,+\infty)(0<\rho_2<\alpha_2<1)$, and $\omega_1, \omega_2, \eta_1, \eta_2 \in C([0,+\infty])$ are nondecreasing functions satisfying the following inequalities:

$$|f_1(t,x(s),x(\lambda_1 s),y(s))| \le \varphi_1(t)\omega_1(\|x\|) + \psi_1(t)\eta_1(\|y\|), \quad |f_2(t,x(s),y(s),y(\lambda_2 s))| \le \varphi_2(t)\omega_2(\|x\|) + \psi_2(t)\eta_2(\|y\|), \tag{28}$$

for all $(x,y) \in PC(J,X)\times PC(J,Y)$, $t\in J$.

(H_5). We suppose that $\varepsilon_1+\varepsilon_2<1$.

Then, problem (3) has at least one solution.

Remark 12. The expressions of ε_1 and ε_2 are given in the proof.

Proof. The set $B_r = \{(x,y)\in PC(J,X)\times PC(J,Y): \|(x,y)\| \le r\}$ is a closed, bounded, and convex set in $PC(J,X)\times PC(J,Y)$ for all $r>0$.

We define the operator T by $T(x,y)(t) = (U(x,y)(t), V(x,y)(t))$ for any $(x,y)\in B_r$ and $t\in[a,b]$, where

$$U(x,y)(t) = \frac{E_{\alpha_1}(-t^{\alpha_1}\gamma_1)}{1+\sigma_1 E_{\alpha_1}(-\gamma_1)}\left(\sum_{k=1}^{m}\frac{I_k(x(t_k))}{E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)} - \sigma_1\int_0^1 (1-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(-(1-s)^{\alpha_1}\gamma_1)f_1(s,x(s), x(\lambda_1 s),y(s))ds\right) - E_{\alpha_1}(-t^{\alpha_1}\gamma_1)\sum_{p=i+1}^{m}\frac{I_p(x(t_p))}{E_{\alpha_1}(-t_p^{\alpha_1}\gamma_1)} + \int_0^t (t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(-(t-s)^{\alpha_1}\gamma_1)f_1(s,x(s), x(\lambda_1 s),y(s))ds, \quad t\in[t_i,t_{i+1}), i=0,1,2,\cdots,m,$$

$$V(x,y)(t) = \frac{E_{\alpha_2}(-t^{\alpha_2}\gamma_2)}{1+\sigma_2 E_{\alpha_2}(-\gamma_2)}\left(\sum_{k=1}^{n}\frac{I_k(y(t_k))}{E_{\alpha_2}(-t_k^{\alpha_2}\gamma_2)} - \sigma_2\int_0^1 (1-s)^{\alpha_2-1}E_{\alpha_2,\alpha_2}(-(1-s)^{\alpha_2}\gamma_2)f_2(s,x(s), y(s),y(\lambda_2 s))ds) - E_{\alpha_2}(-t^{\alpha_2}\gamma_2)\sum_{p=j+1}^{n}\frac{I_p(y(t_p))}{E_{\alpha_2}(-t_p^{\alpha_2}\gamma_2)} + \int_0^t (t-s)^{\alpha_2-1}E_{\alpha_2,\alpha_2}(-(t-s)^{\alpha_2}\gamma_2)f_2(s,x(s), y(s),y(\lambda_2 s))ds, \quad t\in[t_j,t_{j+1}), j=0,1,2,\cdots,n. \tag{29}$$

By splitting the two operators above, we have

$$U_1(x,y)(t)=\int_0^t (t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(-(t-s)^{\alpha_1}\gamma_1)f_1(s,x(s), x(\lambda_1 s),y(s))ds-\frac{\sigma_1 E_{\alpha_1}(-t^{\alpha_1}\gamma_1)}{1+\sigma_1 E_{\alpha_1}(-\gamma_1)}\int_0^1 (1-s)^{\alpha_1-1}\cdot E_{\alpha_1,\alpha_1}(-(1-s)^{\alpha_1}\gamma_1)f_1(s,x(s),x(\lambda_1 s),y(s))ds,$$

$$U_2(x,y)(t)=\frac{E_{\alpha_1}(-t^{\alpha_1}\gamma_1)}{1+\sigma_1 E_{\alpha_1}(-\gamma_1)}\sum_{k=1}^{m}\frac{I_k(x(t_k))}{E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)}-E_{\alpha_1}(-t^{\alpha_1}\gamma_1)\sum_{p=i+1}^{m}\frac{I_p(x(t_p))}{E_{\alpha_1}(-t_p^{\alpha_1}\gamma_1)},$$

$$V_1(x,y)(t)=\int_0^t (t-s)^{\alpha_2-1}E_{\alpha_2,\alpha_2}(-(t-s)^{\alpha_2}\gamma_2)f_2(s,x(s),y(s), y(\lambda_2 s))ds-\frac{\sigma_2 E_{\alpha_2}(-t^{\alpha_2}\gamma_2)}{1+\sigma_2 E_{\alpha_2}(-\gamma_2)}\int_0^1 (1-s)^{\alpha_2-1}E_{\alpha_2,\alpha_2}\cdot(-(1-s)^{\alpha_2}\gamma_2)f_2(s,x(s),y(s),y(\lambda_2 s))ds,$$

$$V_2(x,y)(t)=\frac{E_{\alpha_2}(-t^{\alpha_2}\gamma_2)}{1+\sigma_2 E_{\alpha_2}(-\gamma_2)}\sum_{k=1}^{n}\frac{I_k(y(t_k))}{E_{\alpha_2}(-t_k^{\alpha_2}\gamma_2)}-E_{\alpha_2}(-t^{\alpha_2}\gamma_2)\sum_{p=j+1}^{n}\frac{I_p(y(t_p))}{E_{\alpha_2}(-t_p^{\alpha_2}\gamma_2)}. \quad (30)$$

This upcoming part of the proof requires us to rewrite the operator T as

$$T(x,y)(t)=T_1(x,y)(t)+T_2(x,y)(t), \quad (31)$$

where

$$T_1(x,y)(t)=(U_1(x,y)(t),V_1(x,y)(t)), \quad (32)$$

$$T_2(x,y)(t)=(U_2(x,y)(t),V_2(x,y)(t)). \quad (33)$$

It follows from (H_4) and Holder's inequality that for any $(x,y)\in B_r$ and each $t\in J$, we have

$$\begin{aligned}&\int_0^t \left|(t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(-(t-s)^{\alpha_1}\gamma_1)f_1(s,x(s),x(\lambda_1 s),y(s))\right|ds\\
&\le\frac{1}{\Gamma(\alpha_1)}\int_0^t\left|(t-s)^{\alpha_1-1}\varphi_1(s)\omega_1(r)+\psi_1(s)\eta_1(r)\right|ds\\
&\le\frac{1}{\Gamma(\alpha_1)}\left(\int_0^t\left|(t-s)^{\alpha_1-1}\varphi_1(s)\omega_1(r)\right|+\int_0^t\left|(t-s)^{\alpha_1-1}\psi_1(s)\eta_1(r)\right|ds\right)\\
&\le\frac{1}{\Gamma(\alpha_1)}\left(\int_0^t (t-s)^{(\alpha_1-1)/(1-\rho_1)}ds\right)^{1-\rho_1}\left(\int_0^t(\omega_1(r)\varphi_1(s))^{1/\rho_1}ds\right)^{\rho_1}\\
&\quad+\frac{1}{\Gamma(\alpha_1)}\left(\int_0^t (t-s)^{(\alpha_1-1)/(1-\rho_1)}ds\right)^{1-\rho_1}\left(\int_0^t(\eta_1(r)\psi_1(s))^{1/\rho_1}ds\right)^{\rho_1}\\
&\le\frac{1}{\Gamma(\alpha_1)}\frac{\|\varphi_1\|_{L^{1/\rho_1}(J)}}{((\alpha_1-\rho_1)/(1-\rho_1))^{1-\rho_1}}\omega_1(r)+\frac{1}{\Gamma(\alpha_1)}\frac{\|\psi_1\|_{L^{1/\rho_1}(J)}}{((\alpha_1-\rho_1)/(1-\rho_1))^{1-\rho_1}}\eta_1(r).\end{aligned} \quad (34)$$

By the same method used above, we get

$$\begin{aligned}&\int_0^1\left|(1-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(-(1-s)^{\alpha_1}\gamma_1)f_1(s,x(s),x(\lambda_1 s),y(s))\right|ds\\
&\le\frac{1}{\Gamma(\alpha_1)}\frac{\|\varphi_1\|_{L^{1/\rho_1}(J)}}{((\alpha_1-\rho_1)/(1-\rho_1))^{1-\rho_1}}\omega_1(r)\\
&\quad+\frac{1}{\Gamma(\alpha_1)}\frac{\|\psi_1\|_{L^{1/\rho_1}(J)}}{((\alpha_1-\rho_1)/(1-\rho_1))^{1-\rho_1}}\eta_1(r).\end{aligned} \quad (35)$$

Next, we show that $T(x,y)=U(x,y)+V(x,y)$ is a bounded operator, which means that $T(x,y)\in B_r$, for $(x,y)\in B_r$.

Suppose the opposite; so there exist $(x,y)\in B_r$ and $t\in J$ such that $|U(x,y)|>r/2$ and $|V(x,y)|>r/2$. Assumption (H_2) implies that $|I_i(x(t_i))|\le|I_i(x(t_i))-I_i(0)+I_i(0)|\le C_1 r+K_1^*$ and $|I_j(y(t_j))|\le|I_j(y(t_j))-I_j(0)+I_j(0)|\le C_2 r+K_2^*$.

Thus,

$$\begin{aligned}\frac{r}{2}&<|U(x,y)|\le\frac{\|\varphi_1\|_{L^{1/\rho_1}(J)}}{\Gamma(\alpha_1)((\alpha_1-\rho_1)/(1-\rho_1))^{1-\rho_1}}\omega_1(r)\\
&\quad+\frac{\|\psi_1\|_{L^{1/\rho_1}(J)}}{\Gamma(\alpha_1)((\alpha_1-\rho_1)/(1-\rho_1))^{1-\rho_1}}\eta_1(r)\\
&\quad+\frac{\|\varphi_1\|_{L^{1/\rho_1}(J)}}{\Gamma(\alpha_1)\left|1+\sigma_1E_{\alpha_1}(-\gamma_1)\right|((\alpha_1-\rho_1)/(1-\rho_1))^{1-\rho_1}}\omega_1(r)\\
&\quad+\frac{\|\psi_1\|_{L^{1/\rho_1}(J)}}{\Gamma(\alpha_1)\left|1+\sigma_1E_{\alpha_1}(-\gamma_1)\right|((\alpha_1-\rho_1)/(1-\rho_1))^{1-\rho_1}}\eta_1(r)\\
&\quad+\frac{1}{\left|1+\sigma_1E_{\alpha_1}(-\gamma_1)\right|}\sum_{k=1}^{m}\frac{C_1r+K_1^*}{\left|E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)\right|}+\sum_{k=1}^{m}\frac{C_1r+K_1^*}{\left|E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)\right|}\\
&\le\frac{(2+\sigma_1)}{\left|1+\sigma_1E_{\alpha_1}(-\gamma_1)\right|}\left(\frac{(1+2\sigma_1)\|\varphi_1\|_{L^{1/\rho_1}(J)}}{(2+\sigma_1)\Gamma(\alpha_1)((\alpha_1-\rho_1)/(1-\rho_1))^{1-\rho_1}}\omega_1(r)\right.\\
&\quad+\frac{(1+2\sigma_1)\|\psi_1\|_{L^{1/\rho_1}(J)}}{(2+\sigma_1)\Gamma(\alpha_1)((\alpha_1-\rho_1)/(1-\rho_1))^{1-\rho_1}}\eta_1(r)\\
&\quad\left.+\sum_{k=1}^{m}\frac{C_1r+K_1^*}{\left|E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)\right|}\right).\end{aligned} \quad (36)$$

Dividing both sides by r and taking the lower limit as $r\to+\infty$, we get

$$\begin{aligned}\frac{1}{2}&\le\frac{(2+\sigma_1)}{\left|1+\sigma_1E_{\alpha_1}(-\gamma_1)\right|}\left(\frac{(1+2\sigma_1)\|\varphi_1\|_{L^{1/\rho_1}(J)}}{(2+\sigma_1)\Gamma(\alpha_1)((\alpha_1-\rho_1)/(1-\rho_1))^{1-\rho_1}}\right.\\
&\quad\cdot\lim_{r\to+\infty}\inf\frac{\omega_1}{r}+\frac{(1+2\sigma_1)\|\psi_1\|_{L^{1/\rho_1}(J)}}{(2+\sigma_1)\Gamma(\alpha_1)((\alpha_1-\rho_1)/(1-\rho_1))^{1-\rho_1}}\\
&\quad\left.\cdot\lim_{r\to+\infty}\inf\frac{\eta_1}{r}+\sum_{k=1}^{m}\frac{C_1}{\left|E_{\alpha_1}(-t_k^{\alpha_1}\gamma_1)\right|}\right)=\varepsilon_1.\end{aligned} \quad (37)$$

Similarly, we have

$$\frac{1}{2}\le \frac{(2+\sigma_2)}{\left|1+\sigma_2E_{\alpha_2}(-\gamma_2)\right|}\left(\frac{(1+2\sigma_2)\|\varphi_2\|_{L^{1/\rho_2}(J)}}{(2+\sigma_2)\Gamma(\alpha_2)((\alpha_2-\rho_2)/(1-\rho_2))^{1-\rho_2}}\cdot \liminf_{r\to+\infty}\frac{\omega_2}{r}+\frac{(1+2\sigma_2)\|\psi_2\|_{L^{1/\rho_2}(J)}}{(2+\sigma_2)\Gamma(\alpha_2)((\alpha_2-\rho_2)/(1-\rho_2))^{1-\rho_2}}\cdot \liminf_{r\to+\infty}\frac{\eta_2}{r}+\sum_{k=1}^{n}\frac{C_2}{\left|E_{\alpha_2}\left(-t_k^{\alpha_2}\gamma_2\right)\right|}\right)=\varepsilon_2. \tag{38}$$

So we get $\varepsilon_1+\varepsilon_2\ge 1$, which contradicts (H_5). Hence, $T(x,y)=U(x,y)+V(x,y)$ is a bounded operator, for all $(x,y)\in B_r$.

Now, we show that T_2 is a contraction mapping.

Thus, for all $t\in J$ and $(x,y),(\bar{x},\bar{y})\in PC(J,X)\times PC(J,Y)$, we get

$$\begin{aligned}|U_2(x,y)(t)-U_2(\bar{x},\bar{y})(t)|&\le \frac{\left|E_{\alpha_1}(-t^{\alpha_1}\gamma_1)\right|}{\left|1+\sigma_1E_{\alpha_1}(-\gamma_1)\right|}\cdot\sum_{k=1}^{m}\frac{I_k(x(t_k))-I_k(\bar{x}(t_k))}{\left|E_{\alpha_1}\left(-t_k^{\alpha_1}\gamma_1\right)\right|}+\left|E_{\alpha_1}(-t^{\alpha_1}\gamma_1)\right|\\&\cdot\sum_{k=1}^{m}\frac{I_k(x(t_k))-I_k(\bar{x}(t_k))}{\left|E_{\alpha_1}\left(-t_k^{\alpha_1}\gamma_1\right)\right|}\le\sum_{k=1}^{m}\frac{I_k(x(t_k))-I_k(\bar{x}(t_k))}{\left|E_{\alpha_1}\left(-t_k^{\alpha_1}\gamma_1\right)\right|}\\&\cdot\left(1+\frac{1}{\left|1+\sigma_1E_{\alpha_1}(-\gamma_1)\right|}\right)\le\left(\frac{(2+\sigma_1)}{\left|1+\sigma_1E_{\alpha_1}(-\gamma_1)\right|}\sum_{k=1}^{m}\frac{C_1}{\left|E_{\alpha_1}\left(-t_k^{\alpha_1}\gamma_1\right)\right|}\right)\\&\cdot(\|x-\bar{x}\|+\|y-\bar{y}\|)=\varepsilon_1^*(\|x-\bar{x}\|+\|y-\bar{y}\|).\end{aligned} \tag{39}$$

Similarly, we show that

$$|V_2(x,y)(t)-V_2(\bar{x},\bar{y})(t)|\le\left(\frac{(2+\sigma_2)}{\left|1+\sigma_2E_{\alpha_2}(-\gamma_2)\right|}\sum_{k=1}^{n}\frac{C_2}{\left|E_{\alpha_2}\left(-t_k^{\alpha_2}\gamma_2\right)\right|}\right)\cdot(\|x-\bar{x}\|+\|y-\bar{y}\|)=\varepsilon_2^*(\|x-\bar{x}\|+\|y-\bar{y}\|). \tag{40}$$

It follows from (H_5) that $0<\varepsilon_1^*+\varepsilon_2^*<1$, and

$$|T_2(x,y)(t)-T_2(\bar{x},\bar{y})(t)|\le(\varepsilon_1^*+\varepsilon_2^*)(\|x-\bar{x}\|+\|y-\bar{y}\|). \tag{41}$$

Thus, T_2 is a contraction mapping.

Since the functions f_1 and f_2 are continuous, this implies that the operator T_1 is also continuous. Now, we show that $T_1=(U_1,V_1)$ is compact; we apply the same method as in Theorem 9. One can verify easily that $T_1(B_r)$ is uniformly bounded on $PC(J,X)\times PC(J,Y)$. We now show that $T_1(B_r)$ is equicontinuous on J. Let $f_1^*=\sup_{t\in J}|f_1(t,x(t),x(\lambda_1 t),y(t))|$ and $f_2^*=\sup_{t\in J}|f_2(t,x(t),y(t),y(\lambda_2 t))|$; then, for any $t_i<\xi_2<\xi_1<t_{i+1}$, we have

$$\begin{aligned}|U_1(x,y)(\xi_2)-U_1(x,y)(\xi_1)|&\le\left|\int_0^{\xi_2}(\xi_2-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}\left(-(\xi_2-s)^{\alpha_1}\gamma_1\right)\cdot f_1(s,x(s),x(\lambda_1 s),y(s))ds-\int_0^{\xi_1}(\xi_1-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}\left(-(\xi_1-s)^{\alpha_1}\gamma_1\right)\cdot f_1(s,x(s),x(\lambda_1 s),y(s))ds\right|\\&+\left|\frac{\sigma_1\left(E_{\alpha_1}\left(-\xi_2^{\alpha_1}\gamma_1\right)-E_{\alpha_1}\left(-\xi_1^{\alpha_1}\gamma_1\right)\right)}{1+\sigma_1E_{\alpha_1}(-\gamma_1)}\times\int_0^1(1-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(-(1-s)^{\alpha_1}\gamma_1)f_1(s,x(s),x(\lambda_1 s),y(s))ds\right|\\&\le\left|\int_0^{\xi_2}(\xi_2-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}\left(-(\xi_2-s)^{\alpha_1}\gamma_1\right)f_1(s,x(s),x(\lambda_1 s),y(s))ds\right.\\&-\int_0^{\xi_2}(\xi_1-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}\left(-(\xi_2-s)^{\alpha_1}\gamma_1\right)f_1(s,x(s),x(\lambda_1 s),y(s))ds\\&+\int_0^{\xi_2}(\xi_1-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}\left(-(\xi_2-s)^{\alpha_1}\gamma_1\right)f_1(s,x(s),x(\lambda_1 s),y(s))ds\\&-\int_0^{\xi_2}(\xi_1-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}\left(-(\xi_1-s)^{\alpha_1}\gamma_1\right)f_1(s,x(s),x(\lambda_1 s),y(s))ds\\&\left.-\int_{\xi_2}^{\xi_1}(\xi_1-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}\left(-(\xi_1-s)^{\alpha_1}\gamma_1\right)f_1(s,x(s),x(\lambda_1 s),y(s))ds\right|\\&+\frac{\left|\sigma_1\left(E_{\alpha_1}\left(-\xi_2^{\alpha_1}\gamma_1\right)-E_{\alpha_1}\left(-\xi_1^{\alpha_1}\gamma_1\right)\right)\right|}{\Gamma(\alpha_1)\left|1+\sigma_1E_{\alpha_1}(-\gamma_1)\right|}\int_0^1(1-s)^{\alpha_1-1}|f_1(s,x(s),x(\lambda_1 s),y(s))|ds\\&\le\int_0^{\xi_2}\left|(\xi_2-s)^{\alpha_1-1}-(\xi_1-s)^{\alpha_1-1}\right|\left|E_{\alpha_1,\alpha_1}\left(-(\xi_2-s)^{\alpha_1}\gamma_1\right)\right|f_1^*ds\\&+\int_0^{\xi_2}\left|(\xi_1-s)^{\alpha_1-1}\right|\left|E_{\alpha_1,\alpha_1}\left(-(\xi_2-s)^{\alpha_1}\gamma_1\right)-E_{\alpha_1,\alpha_1}\left(-(\xi_1-s)^{\alpha_1}\gamma_1\right)\right|f_1^*ds\\&+\frac{f_1^*}{\Gamma(\alpha_1)}\left|\int_{\xi_2}^{\xi_1}(\xi_1-s)^{\alpha_1-1}\right|+\frac{\left|\sigma_1\left(E_{\alpha_1}\left(-\xi_2^{\alpha_1}\gamma_1\right)-E_{\alpha_1}\left(-\xi_1^{\alpha_1}\gamma_1\right)\right)\right|}{\Gamma(\alpha_1+1)\left|1+\sigma_1E_{\alpha_1}(-\gamma_1)\right|}f_1^*\\&\le\frac{f_1^*}{\Gamma(\alpha_1)}\int_0^{\xi_2}\left|(\xi_2-s)^{\alpha_1-1}-(\xi_1-s)^{\alpha_1-1}\right|ds+\frac{(\xi_1-\xi_2)^{\alpha_1}f_1^*}{\Gamma(\alpha_1+1)}\\&+\frac{\left|\sigma_1\left(E_{\alpha_1}\left(-\xi_2^{\alpha_1}\gamma_1\right)-E_{\alpha_1}\left(-\xi_1^{\alpha_1}\gamma_1\right)\right)\right|}{\Gamma(\alpha_1+1)\left|1+\sigma_1E_{\alpha_1}(-\gamma_1)\right|}f_1^*+f_1^*\int_0^{\xi_2}\left|(\xi_1-s)^{\alpha_1-1}\right|\\&\cdot\left|E_{\alpha_1,\alpha_1}\left(-(\xi_2-s)^{\alpha_1}\gamma_1\right)-E_{\alpha_1,\alpha_1}\left(-(\xi_1-s)^{\alpha_1}\gamma_1\right)\right|ds\\&\le\frac{(\xi_1-\xi_2)^{\alpha_1}+\xi_1^{\alpha_1}-\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)}f_1^*+\frac{(\xi_1-\xi_2)^{\alpha_1}f_1^*}{\Gamma(\alpha_1+1)}+\frac{\left|\sigma_1\left(E_{\alpha_1}\left(-\xi_2^{\alpha_1}\gamma_1\right)-E_{\alpha_1}\left(-\xi_1^{\alpha_1}\gamma_1\right)\right)\right|}{\Gamma(\alpha_1+1)\left|1+\sigma_1E_{\alpha_1}(-\gamma_1)\right|}f_1^*\\&+f_1^*\int_0^{\xi_2}\left|(\xi_1-s)^{\alpha_1-1}\right|\left|E_{\alpha_1,\alpha_1}\left(-(\xi_2-s)^{\alpha_1}\gamma_1\right)-E_{\alpha_1,\alpha_1}\left(-(\xi_1-s)^{\alpha_1}\gamma_1\right)\right|ds.\end{aligned} \tag{42}$$

By (3) of Lemma 5, it follows that $E_{\alpha_1,\alpha_1}(-t^{\alpha_1}\gamma_1)$ is continuous on $t\in J$, and thus, $E_{\alpha_1,\alpha_1}(-t^{\alpha_1}\gamma_1)$ is uniformly continuous on $t\in J$; hence, for every real number $\varepsilon>0$, there exists a small $\kappa>0$ such that for $t_1,t_2\in J$ with $|t_1-t_2|\le\kappa$, we have

$$\left|E_{\alpha_1,\alpha_1}\left(-t_1^{\alpha_1}\gamma_1\right)-E_{\alpha_1,\alpha_1}\left(-t_2^{\alpha_1}\gamma_1\right)\right|<\frac{\varepsilon}{\xi_2^{\alpha_1/(2-\alpha_1)}}. \tag{43}$$

Let $p=(2-\alpha_1)/2(1-\alpha_1)$ and $q=(2-\alpha_1)/\alpha_1$. Then, $p>1$, $q>1$, and $(1/p)+(1/q)=1$. Applying Holder's inequality yields

$$\begin{aligned}&\int_0^{\xi_2}(\xi_1-s)^{\alpha_1-1}\left|E_{\alpha_1,\alpha_1}\left(-(\xi_2-s)^{\alpha_1}\gamma_1\right)-E_{\alpha_1,\alpha_1}\left(-(\xi_1-s)^{\alpha_1}\gamma_1\right)\right|ds\\&\quad\le\left[\int_0^{\xi_2}(\xi_1-s)^{(\alpha_1-1)(2-\alpha_1)/2(1-\alpha_1)}ds\right]^{2(1-\alpha_1)/(2-\alpha_1)}\end{aligned}$$

$$\times \left[\int_0^{\xi_2} \left(E_{\alpha_1,\alpha_1}\left(-(\xi_2 - s)^{\alpha_1}\gamma_1\right) - E_{\alpha_1,\alpha_1}\left(-(\xi_1 - s)^{\alpha_1}\gamma_1\right)\right)^{(2-\alpha_1)/\alpha_1} ds\right]^{\alpha_1/(2-\alpha_1)}$$
$$\leq \left[\frac{2\xi_1^{\alpha_1/2} - 2(\xi_1 - \xi_2)^{\alpha_1/2}}{\alpha_1}\right]^{2(1-\alpha_1)/(2-\alpha_1)} \varepsilon. \tag{44}$$

Hence,

$$\begin{aligned}|U_1(x,y)(\xi_2) - U_1(x,y)(\xi_1)| &\leq \frac{(\xi_1 - \xi_2)^{\alpha_1} + \xi_1^{\alpha_1} - \xi_2^{\alpha_1}}{\Gamma(\alpha_1 + 1)} f_1^* \\ &+ \frac{(\xi_1 - \xi_2)^{\alpha_1} f_1^*}{\Gamma(\alpha_1 + 1)} + \frac{\left|\sigma_1\left(E_{\alpha_1}\left(-\xi_2^{\alpha_1}\gamma_1\right) - E_{\alpha_1}\left(-\xi_1^{\alpha_1}\gamma_1\right)\right)\right|}{\Gamma(\alpha_1 + 1)\left|1 + \sigma_1 E_{\alpha_1}(-\gamma_1)\right|} f_1^* \\ &+ f_1^* \left[\frac{2\xi_1^{\alpha_1/2} - 2(\xi_1 - \xi_2)^{\alpha_1/2}}{\alpha_1}\right]^{2(1-\alpha_1)/(2-\alpha_1)} \varepsilon \longrightarrow 0,\end{aligned} \tag{45}$$

as $\xi_2 \longrightarrow \xi_1$, which implies that U_1 is equicontinuous on J. Similarly, the operator V_1 is also equicontinuous on J. Hence, $T_1(B_r)$ is relatively compact on J. It follows by Arzela-Ascoli's theorem that T_1 is compact. Therefore, we conclude from Lemma 6 that problem (3) has at least one solution.

4. Examples

Example 1. Consider the following coupled fractional pantograph differential equations with instantaneous impulses given by

$$\begin{cases} {}^cD^{1/2}x(t) + x(t) = \dfrac{\cos(t)}{(t+8)^2}\dfrac{x^2(t)}{1+x^2(t)} + \dfrac{\sin(t)}{(e^t+7)^2}x\left(\dfrac{1}{3}t\right) + \dfrac{\cos(t)}{(t+8)^2}\dfrac{y^2(t)}{1+y^2(t)}, & t \in J \setminus \left\{\dfrac{1}{2}\right\}, \\ {}^cD^{1/2}y(t) + y(t) = \dfrac{\sin(t)}{(t+8)^2}\dfrac{x^2(t)}{1+x^2(t)} + \dfrac{\cos(t)}{(t+7)^2}\dfrac{y^2(t)}{1+y^2(t)} + \dfrac{\sin(t)}{(e^t+8)}y\left(\dfrac{1}{3}t\right), & t \in J \setminus \left\{\dfrac{1}{2}\right\}, \\ \Delta x_{|t=1/2}(0) = \dfrac{|x(1/2)|}{40 + |x(1/2)|}, \\ \Delta y_{|t=1/2}(0) = \dfrac{|y(1/2)|}{40 + |y(1/2)|}, \\ x(0) = -x(1), \\ y(0) = -y(1). \end{cases} \tag{46}$$

By comparing (46) with problem (3), we get

$$\begin{aligned} f_1(t, x(t), x(\lambda_1 t), y(t)) &= \frac{\cos(t)}{(t+8)^2}\frac{x^2(t)}{1+x^2(t)} + \frac{\sin(t)}{(e^t+7)^2}x\left(\frac{1}{3}t\right) \\ &+ \frac{\cos(t)}{(t+8)^2}\frac{y^2(t)}{1+y^2(t)}, \\ f_2(t, x(t), y(t), y(\lambda_2 t)) &= \frac{\sin(t)}{(t+8)^2}\frac{x^2(t)}{1+x^2(t)} + \frac{\cos(t)}{(t+7)^2}\frac{y^2(t)}{1+y^2(t)} \\ &+ \frac{\sin(t)}{(e^t+8)^2}y\left(\frac{1}{3}t\right). \end{aligned} \tag{47}$$

We have $a_1 = a_2 = b_1 = b_2 = \sigma_1 = \sigma_2 = 1$, $\alpha_1 = \alpha_2 = 1$, and $\gamma_1 = \gamma_2 = 1$.

$$\begin{aligned} I_i(x) &= \frac{|x(t)|}{40 + |x(t)|}, \\ I_i(y) &= \frac{|y(t)|}{40 + |y(t)|}. \end{aligned} \tag{48}$$

Then, for any $x, \bar{x}, y, \bar{y} \in \mathbb{R}$ and $t \in J$, we have

$$\begin{aligned}&|f_1(t, x(t), x(\lambda_1 t), y(t)) - f_1(t, \bar{x}(t), \bar{x}(\lambda_1 t), y(t))| \\ &\quad\leq \frac{1}{32}(\|x - \bar{x}\| + \|y - \bar{y}\|),\end{aligned}$$

$$\begin{aligned}&|f_2(t, x(t), y(t), y(\lambda_2 t)) - f_2(t, \bar{x}(t), \bar{y}(t), \bar{y}(\lambda_2 t))| \\ &\quad\leq \frac{1}{32}(\|x - \bar{x}\| + \|y - \bar{y}\|),\end{aligned}$$

$$|I_i(x) - I_i(\bar{x}| \leq \frac{1}{40}(\|x - \bar{x}\| + \|y - \bar{y}\|),$$

$$\left|I_j(y) - I_j(\bar{y}\right| \leq \frac{1}{40}(\|x - \bar{x}\| + \|y - \bar{y}\|). \tag{49}$$

Then, by simple calculations, we can easily see that

$$\begin{aligned} L_1 = L_2 &= \frac{1}{32}, \\ C_1 = C_2 &= \frac{1}{40}, \\ E_{1/2}(-1) &\approx 0.42, \\ E_{1/2}\left(\left(-\frac{1}{2}\right)^{1/2}\right) &\approx 0.52, \\ \Gamma\left(\frac{3}{2}\right) &\approx 0.89, \\ \mu_1 + \mu_2 = 2 \times \frac{3}{|1 + E_{1/2}(-1)|}&\left(\sum_{k=1}^{n}\frac{C_1}{E_{1/2}\left(-(1/2)^{1/2}\right)} + \frac{12L_1}{3\Gamma(3/2)}\right) \approx 0.796 < 1. \end{aligned} \tag{50}$$

Therefore, all the hypotheses of Theorem 9 are satisfied. Hence, problem (46) has a unique solution (x, y) on $[0, 1]$.

Example 2. Consider the following impulsive fractional coupled equations with antiperiodic conditions given by

$$\begin{cases} {}^cD^{1/2}x(t) + 2x(t) = \dfrac{\sqrt[3]{t+1}\sin(2t)}{14}\left(\dfrac{|x(t)|}{1+|x(t)|}\right) + \dfrac{\sqrt[3]{t+1}}{(7e^t)^2}x\left(\dfrac{3}{2}t\right) + \dfrac{\sqrt[3]{t+1}\cos(2t)}{14}\left(\dfrac{|y(t)|}{1+|y(t)|}\right), & t \in j \setminus \left\{\dfrac{1}{2}\right\}, \\ {}^cD^{1/2}y(t) + 2y(t) = \dfrac{\sqrt[3]{t+1}\cos(2t)}{14}\left(\dfrac{|x(t)|}{1+|x(t)|}\right) + \dfrac{\sqrt[3]{t+1}\sin(2t)}{14}\left(\dfrac{|y(t)|}{1+|y(t)|}\right) + \dfrac{\sqrt[3]{t+1}}{(7e^t)^2}y\left(\dfrac{3}{2}t\right), \\ \Delta x_{|t=1/2}(0) = \dfrac{|x(1/2)|}{90+|x(1/2)|}, \\ \Delta y_{|t=1/2}(0) = \dfrac{|y(1/2)|}{90+|y(1/2)|}, \\ x(0) + x(1) = 0, \\ y(0) + y(1) = 0. \end{cases} \tag{51}$$

By comparing (51) with problem (3), we have

$$\begin{aligned} f_1(t, x(t), x(\lambda_1 t), y(t)) &= \frac{\sqrt[3]{t+1}\sin(2t)}{14}\left(\frac{|x(t)|}{1+|x(t)|}\right) + \frac{\sqrt[3]{t+1}}{(7e^t)^2}x\left(\frac{3}{2}t\right) \\ &\quad + \frac{\sqrt[3]{t+1}\cos(2t)}{14}\left(\frac{|y(t)|}{1+|y(t)|}\right), \\ f_2(t, x(t), y(t), y(\lambda_2 t)) &= \frac{\sqrt[3]{t+1}\cos(2t)}{14}\left(\frac{|x(t)|}{1+|x(t)|}\right) + \frac{\sqrt[3]{t+1}\sin(2t)}{14} \\ &\quad \cdot \left(\frac{|y(t)|}{1+|y(t)|}\right) + \frac{\sqrt[3]{t+1}}{(7e^t)^2}y\left(\frac{3}{2}t\right). \end{aligned} \tag{52}$$

This implies that

$$\begin{aligned} |f_1(t, x(t), x(\lambda_1 t), y(t))| &\le \left(\frac{\sqrt[3]{t+1}\sin(2t)}{14}\right)(\|x\|+1) \\ &\quad + \left(\frac{\sqrt[3]{t+1}\sin(2t)}{14}\right)\|y\|, \\ |f_2(t, x(t), y(t), y(\lambda_2 t))| &\le \left(\frac{\sqrt[3]{t+1}\sin(2t)}{14}\right)\|x\| \\ &\quad + \left(\frac{\sqrt[3]{t+1}\sin(2t)}{14}\right)(\|y\|+1), \end{aligned} \tag{53}$$

for any $x, \bar{x}, y, \bar{y} \in \mathbb{R}$ and $t \in J$.

Thus, applying the same procedure as in Example 1, we get

$$\begin{aligned} C_1 &= C_2 = \frac{1}{90}, \\ E_{1/2}(-2) &\approx 0.25, \\ E_{1/2}\left(-2\left(\frac{1}{2}\right)^{1/2}\right) &\approx 0.40, \\ \Gamma\left(\frac{1}{2}\right) &\approx 3.14, \\ \varphi_1(t) = \varphi_2(t) = \psi_1(t) = \psi_2(t) &= \frac{\sqrt[3]{t+1}}{14}. \end{aligned} \tag{54}$$

This implies that

$$\begin{aligned} \lim_{r\to+\infty}\inf\frac{\omega_1}{r} &= \lim_{r\to+\infty}\inf\frac{\eta_1}{r} \\ &= \lim_{r\to+\infty}\inf\frac{\omega_2}{r} \\ &= \lim_{r\to+\infty}\inf\frac{\eta_2}{r} = 1, \end{aligned} \tag{55}$$

$$\varepsilon_1 + \varepsilon_2 \approx 0.84 < 1. \tag{56}$$

Thus, all the hypotheses of Theorem 11 are satisfied. Hence, problem (51) has at least a solution (x, y) on $[0, 1]$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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