Research Article

Reparameterization Invariant Model of a Supersymmetric System: BRST and Supervariable Approaches

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We carry out the Becchi-Rouet-Stora-Tyutin (BRST) quantization of the one $(0 + 1)$-dimensional (1D) model of a free massive spinning relativistic particle (i.e., a supersymmetric system) by exploiting its *classical* infinitesimal and continuous reparameterization symmetry transformations. We use the *modified* Bonora-Tonin (BT) supervariable approach (MBTSA) to BRST formalism to obtain the nilpotent (anti-)BRST symmetry transformations of the target space variables and the (anti-)BRST invariant Curci-Ferrari- (CF-) type restriction for the 1D model of our supersymmetric (SUSY) system. The nilpotent (anti-)BRST symmetry transformations for *other* variables of our model are derived by using the (anti-)chiral supervariable approach (ACSA) to BRST formalism. Within the framework of the *latter*, we have shown the existence of the CF-type restriction by proving the (i) symmetry invariance of the coupled Lagrangians and (ii) the absolute anticommutativity property of the conserved (anti-)BRST charges. The application of the MBTSA to a *physical* SUSY system (i.e., a 1D model of a massive spinning particle) is a *novel* result in our present endeavor. In the application of ACSA, we have considered *only* the (anti-)chiral super expansions of the supervariables. Hence, the observation of the absolute anticommutativity of the (anti-)BRST charges is a novel result. The CF-type restriction is *universal* in nature as it turns out to be the *same* for the SUSY and non-SUSY reparameterization (i.e., 1D diffeomorphism) invariant models of the (non-)relativistic particles.

1. Introduction

The Becchi-Rouet-Stora-Tyutin (BRST) quantization scheme is one of the most elegant approaches to quantize the locally gauge *and* diffeomorphism invariant theories where the local *classical* transformation parameters are traded with the (anti-)ghost fields at the *quantum* level [1–4]. For the quantization of the *classical* supersymmetric gauge theories (with the bosonic and fermionic transformation parameters), the BRST quantization scheme requires the fermionic as well as the bosonic (anti-)ghost fields/variables (see, e.g., [5, 6]). Some of the key characteristic features of the BRST quantization scheme are (i) for a given *local* gauge and/or diffeomorphism symmetry, there exist *two* nilpotent symmetries which are christened as the BRST and anti-BRST symmetries, (ii) the (anti-)BRST symmetries ($s_{(a)b}$) are fermionic (i.e., nilpotent) and absolutely anticom-

muting (i.e., $s_b s_{ab} + s_{ab} s_b = 0$) in nature, (iii) the quantum gauge (i.e., BRST) invariance and unitarity are respected *together* in the perturbative computations at any arbitrary order, (iv) there is appearance of the Curci-Ferrari- (CF-) type restriction(s) which are responsible for the absolute anticommutativity of the (anti-)BRST transformations *and* they lead to the existence of the coupled (but equivalent) Lagrangian (densities) for the BRST *quantized* theory which respect *both* (i.e., BRST and anti-BRST) nilpotent *quantum* symmetries, and (v) the (anti-)BRST symmetries transform a bosonic field/variable to a fermionic field/variable and vice-versa. Hence, these symmetries are supersymmetric type (i.e., SUSY type).

Physically, the nilpotency property of the (anti-)BRST symmetry transformations corresponds to the *fermionic* nature of these *quantum* symmetries *and* the absolute anticommutativity property encodes the linear independence of

the BRST and anti-BRST symmetry transformations. The former property encodes the SUSY-type transformations. As pointed out earlier, the absolute anticommutativity property *owes* its dependence on the existence of the CF-type restriction(s). The BRST approach to Abelian 1-form gauge theory is an *exception* where the CF-type restriction is *trivial* (but it turns out to be the limiting case of the non-Abelian 1-form theory which is endowed with the CF condition [7]). It is the usual superfield approach (USFA) to BRST formalism [8–15] which provides the interpretation and origin for the abstract mathematical properties (i.e., nilpotency and anticommutativity) that are associated with the (anti-)BRST symmetries. Furthermore, the USFA leads to the derivation of the CF condition [7] in the context of a 4D 1-form non-Abelian theory (see, e.g., [10–12]) which is found to be an (anti-)BRST invariant quantity. Hence, *it is a physical* restriction on the BRST *quantized* theory.

The USFA to BRST formalism [8–15] leads to the derivation of *only* the off-shell nilpotent symmetries that are associated with the gauge and (anti-)ghost fields/variables. It does *not* shed *any* light on the (anti-)BRST transformations that are associated with the *matter* fields in an *interacting* gauge theory. There have been consistent extensions of the USFA (see, e.g., [16–18]) where additional *quantum* gauge invariant restrictions on the superfields have been imposed to derive the (anti-)BRST symmetry transformations for the gauge, (anti-)ghost, and *matter* fields *together*. This extended version of the superfield approach to BRST formalism has been called as the augmented version of superfield approach (AVSA). In our recent works (see, e.g., [19–21]), we have been able to develop a *simpler* off-shoot of AVSA where *only* the (anti-)chiral superfields/supervariables have been taken into account. The *quantum* gauge (i.e., (anti-)BRST) invariant restrictions on these (anti-)chiral superfields/supervariables have led to the deduction of (anti-)BRST symmetry transformations for *all* the fields/variables. This approach to BRST formalism has been named as the (anti-)chiral superfield/supervariable approach (ACSA) to BRST formalism where the existence of the CF-type restriction(s) has been shown by proving (i) the absolute anticommutativity of the (anti-)BRST charges and (ii) the invariance of the coupled (but equivalent) Lagrangian (densities) of the (anti-)BRST invariant theories.

It has been a challenging problem to apply the superfield approach to BRST formalism in the context of (super)string and gravitational theories which are diffeomorphism invariant. In a recent paper [22], it has been proposed that the diffeomorphism symmetry can be taken into consideration within the framework of superfield approach to BRST formalism. This approach to BRST formalism has been called as the *modified* Bonora-Tonin superfield/supervariable approach (MBTSA) to BRST formalism which has been recently applied to the physical system of a 1D scalar (non-)relativistic particles [23, 24]. To be precise, judicious combination of MBTSA and ACSA has been very fruitful in our recent works [23, 24] where we have been able to derive the proper (anti-)BRST symmetries for *all* the quantum variables along with the CF-type restriction in a systematic fashion. In the proposal by Bonora and Malik [25], the

diffeomorphism symmetry transformations have been incorporated into the superfields which are defined on a $(D, 2)$-dimensional supermanifold on which a D-dimensional *ordinary* diffeomorphism invariant theory is generalized. We have exploited the mathematical rigor and beauty of the MBTSA in our present endeavor for the BRST analysis as well as quantization of a 1D diffeomorphism invariant SUSY system.

To be precise, in our present investigation, we have applied the theoretical beauty and strength of MBTSA to derive the off-shell nilpotent (anti-)BRST transformations for the target space canonically conjugate *bosonic* variables (x_μ and p^μ) and *fermionic* variables (ψ_μ, ψ_5) along with the (anti-)BRST invariant CF-type restrictions: $B + \bar{B} + i(\bar{C}\dot{C} - \dot{\bar{C}}C) = 0$ which is responsible for (i) the validity (cf. Equation (22)) of the absolute anticommutativity (i.e., $\{s_b, s_{ab}\} = s_b s_{ab} + s_{ab} s_b = 0$) of the off-shell nilpotent ($s_{(a)b}^2 = 0$) (anti-)BRST symmetry transformations $s_{(a)b}$ and (ii) the derivation of L_B and $L_{\bar{B}}$ which are coupled (but equivalent) (cf. Equation (24)). The proper (anti-)BRST transformations of the *rest* of the variables have been derived by using the ACSA to BRST formalism. It is worth pointing out that, in the case of MBTSA, we have taken into account the *full* super expansions of the supervariables along *all* the possible Grassmannian directions of the *general* $(1, 2)$-dimensional supermanifold. On the other hand, we have performed *only* the (anti-)chiral super expansions of the supervariables in the context of ACSA to BRST formalism. We have derived the *exact* expression for the CF-type restriction by demanding (i) the invariance of the coupled (but equivalent) Lagrangians and (ii) the validity of the absolute anticommutativity of the off-shell nilpotent (anti-)BRST symmetries as well as conserved (anti-)BRST charges in the *ordinary* space *and* in the *superspace* (within the framework of ACSA to BRST formalism). These derivations of the CF-type restrictions, by various theoretical methods, are *novel* results in our present investigation.

The following key factors have been at the heart of our curiosity to pursue our present investigation. First, we have been able to apply the MBTSA to a reparameterization invariant model of the *scalar* relativistic particle to derive the (anti-)BRST symmetry transformations as well as the (anti-)BRST invariant CF type of restriction [24]. Thus, it has been very important for us to apply the *same* mathematical technique (i.e., MBTSA) to a *physically* interesting SUSY model of a reparameterization invariant theory where the fermionic as well as bosonic variables exist. Second, we have been very curious to verify the *universality* of the CF-type restriction in the context of BRST quantization of the 1D diffeomorphism (i.e., reparameterization) invariant theories. We find that the nature and form of the CF-type restriction are the *same* for the SUSY as well as non-SUSY theories. Third, it has been very interesting to note that the gauge-fixing and Faddeev-Popov ghost terms *together* are the same for the reparameterization invariant *scalar* and *SUSY* relativistic as well as a nonrelativistic particle [23]. Finally, our present investigation (as well as others [23, 24]) is our

modest initial steps to apply the MBTSA as well as the ACSA to BRST formalism *together* to physically interesting 4D (and higher dimensional) diffeomorphism invariant theories which are important from the point of view of the modern developments in gravitational as well as (super)string theories (and related extended objects) of the theoretical high energy physics.

The theoretical contents of our present endeavor are organised as follows. In Section 2, we discuss a couple of continuous and infinitesimal symmetry transformations and establish their relationship with the infinitesimal and continuous 1D diffeomorphism (i.e., reparameterization) symmetry transformations. Our Section 3 is devoted to the upgradation of the *classical* infinitesimal reparameterization symmetry transformations to the *quantum* off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations. The *latter* property is satisfied due to the existence of an (anti-)BRST invariant CF-type restriction. This section *also* contains the coupled (but equivalent) Lagrangians that respect *both* the (anti-)BRST symmetry transformations on the submanifold (of the subspace of *quantum* variables) where the CF-type restriction is satisfied. In Section 4, we derive the (anti-)BRST transformations for the target space fermionic as well as bosonic variables. In addition, we deduce the CF type of restriction by exploiting the theoretical strength of MBTSA. We utilize the potential of ACSA to BRST formalism to obtain the nilpotent (anti-)BRST symmetry transformations for the *other* variables of our BRST-invariant theory in Section 5. We capture the (anti-)BRST invariance of the coupled (but equivalent) Lagrangians within the ambit of ACSA and establish, once again, the existence of our *quantum* (anti-)BRST invariant CF-type restriction in Section 6. Section 7 of our present endeavor contains theoretical proof of the nilpotency and absolute anticommutativity of the conserved and off-shell nilpotent (anti-)BRST charges. Finally, in Section 8, we summarize our key accomplishments and point out a few directions for further investigations.

In our Appendices A, B, and C, we collect some explicit computations that corroborate a few key claims and statements that have been made in the main body of the text of our present endeavor. Our Appendix D is devoted to provide an alternative proof of the existence of an (anti-)BRST invariant CF-type restriction on our theory.

Convention and notations: we take the flat metric tensor of the D-dimensional target spacetime manifold as $\eta_{\mu\nu} = (+1, -1, -1, -1 \cdots)$ so that the dot product between *two* nonnull vectors (P_μ, Q_μ) is $P \cdot Q = \eta_{\mu\nu} P^\mu Q^\nu = P_0 Q_0 - P_i Q_i$ where the Greek indices $\mu, \nu, \lambda \cdots = 0, 1, 2 \cdots D - 1$ and Latin indices $i, j, k \cdots = 1, 2, 3 \cdots D - 1$. We adopt the convention of the left-derivative w.r.t. *all* the fermionic variables. We *always* denote the nilpotent (anti-)BRST transformations by the notations $s_{(a)b}$. Our 1D model is generalized onto a $(1, 2)$-dimensional supermanifold which is parameterized by the superspace coordinates $Z^M = (\tau, \theta, \overline{\theta})$ where τ is the bosonic evolution parameter and a pair of Grassmannian variable $(\theta, \overline{\theta})$ satisfy $\theta^2 = \overline{\theta}^2 = 0, \theta \overline{\theta} + \overline{\theta} \theta = 0$. In our present investigation, we shall focus *only* on the $(1, 1)$-

-dimensional (anti-)chiral super submanifolds of our chosen *general* $(1, 2)$-dimensional supermanifold in the context of ACSA.

2. Preliminaries: Continuous and Infinitesimal Reparameterization Symmetry Transformations

We divide our present section into two parts. We discuss, in our Section 2.1, a few *classical* infinitesimal continuous symmetries and their relationships with the *classical* infinitesimal reparameterization symmetry transformations. Our Section 2.2 is devoted to a concise description of the *quantum* (anti-)BRST symmetry transformations.

2.1. Some Classical Infinitesimal and Continuous Symmetries. We begin with the following *first-order* Lagrangian (L_f) for the free one $(0 + 1)$-dimensional (1D) massive spinning (i.e., SUSY) relativistic particle (see, e.g., [5, 26])

$$L_f = p_\mu \dot{x}^\mu - \frac{e}{2}\left(p^2 - m^2\right) + \frac{i}{2}\left(\psi_\mu \dot{\psi}^\mu - \psi_5 \dot{\psi}_5\right) + i \chi \left(p_\mu \psi^\mu - m\psi_5\right),$$

$$(1)$$

where we have parametrized the trajectory of the particle by τ and defined the "generalized" velocities $(\dot{x}^\mu = dx^\mu/d\tau, \dot{\psi}^\mu = d\psi^\mu/d\tau)$ w.r.t. *it*. This 1D trajectory is embedded in the D-dimensional flat Minkowskian target spacetime manifold where (x_μ, p^μ) (with $\mu = 0, 1, 2, \cdots D - 1$) are the canonical conjugate pair of spacetime and momenta variables which are function of the evolution parameter τ. We have the fermionic variables (χ, ψ_μ, ψ_5) in our theory. The Lagrangian (1) *also* has the bosonic variable e and fermionic variable χ as the Lagrange multiplier variables. These *latter* variables behave like the "gauge" and "superaguge" variables due to their transformation properties under the gauge and supersymmetric gauge transformations. In fact, the fermionic variable ψ_μ is the superpartner of x_μ and the other fermionic variable ψ_5 has been introduced to incorporate the rest mass m into the Lagrangian L_f where the mass-shell condition $(p^2 - m^2 = 0)$ is satisfied by the free $(\dot{p}_\mu = 0)$ massive spinning relativistic particle. Our present 1D model is interesting because it provides a prototype example of a supersymmetric (SUSY) gauge theory and its generalization to the 4D theory provides an example of the supergravity theory where fermionic variable ψ_μ becomes the Rarita-Schwinger Lorentz vector spin 3/2 field and the einbein variable e turns *itself* into the vierbein field.

The action integral $S = \int_{-\infty}^{+\infty} d\tau L_f$ respects the following infinitesimal and continuous reparameterization symmetry

transformations (δ_r):

$$\delta_r x_\mu = \varepsilon \dot{x}_\mu,$$
$$\delta_r \psi_\mu = \varepsilon \dot{\psi}_\mu,$$
$$\delta_r p_\mu = \varepsilon \dot{p}_\mu,$$
$$\delta_r \psi_5 = \varepsilon \dot{\psi}_5, \tag{2}$$
$$\delta_r \chi = \frac{d}{d\tau}(\varepsilon \chi),$$
$$\delta_r e = \frac{d}{d\tau}(\varepsilon e),$$

because the first-order Lagrangian (L_f) transforms, under the above infinitesimal reparameterization symmetry transformation (δ_r), as

$$\delta_r L_f = \frac{d}{d\tau}\left[\varepsilon L_f\right] \Longrightarrow \delta_r S = 0, \tag{3}$$

where δ_r *basically* corresponds to the infinitesimal 1D diffeomorphism/reparameterization transformation: $\tau \longrightarrow \tau' = \tau - \varepsilon(\tau)$. Here, the transformation parameter $\varepsilon(\tau)$ is infinitesimal. It is an elementary exercise to note that, if we set all the fermionic variables equal to zero (i.e., $\chi, \psi_\mu, \psi_5 = 0$), we obtain an infinitesimal gauge symmetry transformation (δ_g) from the infinitesimal reparameterization symmetry transformation (2) as follows:

$$\delta_g x_\mu = \xi p_\mu,$$
$$\delta_g p_\mu = 0,$$
$$\delta_g e = \frac{d}{d\tau}(\xi) = \dot{\xi}, \tag{4}$$
$$\delta_g \psi_\mu = \delta_g \psi_5 = \delta_g \chi = 0,$$

where we have identified $e\varepsilon = \xi$ and used the Eular-Lagrange eqations of motions (EL-EOMs): $\dot{x}_\mu = ep_\mu, \dot{p}_\mu = 0$. In Equation (4), the bosonic infinitesimal transformation parameter $\xi(\tau)$ is nothing but the gauge transformation parameter. It can be readily checked that we have the following transformation for L_f and S under δ_g:

$$\delta_g L_f = \frac{d}{d\tau}\left[\frac{\xi}{2}(p^2 + m^2)\right] \Longrightarrow \delta_g S = 0, S = \int_{-\infty}^{+\infty} d\tau L_f. \tag{5}$$

We have infinitesimal classical supergauge symmetry transformations (δ_{sg}) in our theory which transforms the *fermionic* variables into their *bosonic* counterparts and vice-versa. These continuous and infinitesimal symmetry transformations are

$$\delta_{sg} x_\mu = \kappa \psi_\mu,$$
$$\delta_{sg} p_\mu = 0,$$
$$\delta_{sg} \psi_\mu = i\kappa p_\mu,$$
$$\delta_{sg} \psi_5 = i\kappa m, \tag{6}$$
$$\delta_{sg} \chi = i\dot{\kappa},$$
$$\delta_{sg} e = 2\kappa \chi,$$

where the infinitesimal transformation parameter $\kappa(\tau)$ is *fermionic* (i.e., $\kappa^2 = 0$) in nature. It is straightforward to observe that we have the following:

$$\delta_{sg} L_f = \frac{d}{d\tau}\left[\frac{\kappa}{2}\left(p_\mu \psi^\mu + m\psi_5\right)\right] \Longrightarrow \delta_{sg} S = 0. \tag{7}$$

Under the combined $\delta = \delta_g + \delta_{sg}$ *classical* symmetry transformation (δ), we have the following continuous and infinitesimal symmetry transformations (δ), namely,

$$\delta x_\mu = \xi p_\mu + \kappa \psi_\mu,$$
$$\delta \psi_\mu = i\kappa p_\mu,$$
$$\delta p_\mu = 0,$$
$$\delta e = \dot{\xi} + 2\kappa\chi, \tag{8}$$
$$\delta \chi = i\dot{\kappa},$$
$$\delta \psi_5 = i\kappa m,$$

which lead to the transformation of the first-order Lagrangian L_f as

$$\delta L_f = \frac{d}{d\tau}\left[\frac{\xi}{2}(p^2 + m^2) + \frac{\kappa}{2}\left(p_\mu \psi^\mu + m\psi_5\right)\right] \Longrightarrow \delta S = 0. \tag{9}$$

Thus, the continuous and infinitesimal transformation δ is indeed a *symmetry* transformation for the action integral $S = \int_{-\infty}^{\infty} d\tau L_f$ due to Gauss's divergence theorem.

As the gauge symmetry transformation (4) can be incorporated into the reparameterization symmetry transformations (2) with the help of some EL-EOMs and some identification of the transformation parameters, in exactly similar fashion, the combined (super)gauge symmetry transformations (8) can be incorporated into the reparameterization symmetry transformations (2) if we take the help of the following EL-EOMs, namely,

$$\dot{p}_\mu = 0,$$
$$\dot{x}_\mu = ep_\mu - i\chi\psi_\mu,$$
$$\dot{\psi}_\mu = \chi p_\mu, \tag{10}$$
$$\dot{\psi}_5 = m\chi,$$

and identify the transformation parameters as $e\varepsilon = \xi$ and $-$

$i \varepsilon \chi = \kappa$ (see, e.g., [5, 26] for details). Thus, we note that the *classical* infinitesimal reparameterization symmetry transformations (2) are a set of very *general* kind of symmetry transformations whose *special* cases are the continuous and infinitesimal symmetry transformations (4) and (8).

2.2. Quantum (Anti-)BRST Symmetries Corresponding to the Classical (Super)gauge Symmetry Transformations.

The *classical* continuous and infinitesimal (super)gauge symmetry transformations (8) can be elevated to their counterpart *quantum* nilpotent ($s_{(a)b}^2 = 0$), infinitesimal, and continuous (anti-)BRST transformations $s_{(a)b}$ as follows [5, 6, 26]:

$$s_{ab} x_\mu = \bar{c} p_\mu + \bar{\beta} \psi_\mu,$$
$$s_{ab} \psi_\mu = i \bar{\beta} p_\mu,$$
$$s_{ab} e = \dot{\bar{c}} + 2 \bar{\beta} \chi,$$
$$s_{ab} c = i \bar{b},$$
$$s_{ab} \bar{c} = -i \bar{\beta}^2,$$
$$s_{ab} p_\mu = 0,$$
$$s_{ab} \bar{\beta} = 0, \qquad (11)$$
$$s_{ab} \beta = -i \gamma,$$
$$s_{ab} \gamma = 0,$$
$$s_{ab} \chi = i \dot{\bar{\beta}},$$
$$s_{ab} b = 2 i \bar{\beta} \gamma,$$
$$s_{ab} \psi_5 = i \bar{\beta} m,$$
$$s_{ab} \bar{b} = 0,$$

$$s_b x_\mu = c p_\mu + \beta \psi_\mu,$$
$$s_b \psi_\mu = i \beta p_\mu,$$
$$s_b e = \dot{c} + 2 \beta \chi,$$
$$s_b p_\mu = 0,$$
$$s_b c = -i \beta^2,$$
$$s_b \beta = 0,$$
$$s_b \bar{c} = i b, \qquad (12)$$
$$s_b \bar{\beta} = i \gamma,$$
$$s_b \gamma = 0,$$
$$s_b \chi = i \dot{\beta},$$
$$s_b \bar{b} = -2 i \beta \gamma,$$
$$s_b b = 0,$$
$$s_b \psi_5 = i \beta m,$$

where the *fermionic* ($c^2 = \bar{c}^2 = c\bar{c} + \bar{c}c = 0$) (anti-)ghost variables ($\bar{c}$)c and the *bosonic* ($\beta^2 = \bar{\beta}^2 \neq 0$, $\beta\bar{\beta} = \bar{\beta}\beta$) (anti-)ghost variables ($\bar{\beta}$)β correspond to the bosonic and fermionic gauge and supergauge transformation parameters ξ and κ of Equation (8), respectively. The variables ($\bar{b}$)b are the Nakanishi-Lautrup-type auxiliary variables, and γ is an additional fermionic ($\gamma^2 = 0$) variable in our BRST-quantized (as well as invariant) theory.

It is straightforward to note that the anticommutativity property of the off-shell nilpotent ($s_{(a)b}^2 = 0$) (anti-)BRST symmetry transformations ($s_{(a)b}$), namely,

$$\{s_b, s_{ab}\} x_\mu = i \left(b + \bar{b} + 2 \beta \bar{\beta}\right) p_\mu,$$
$$\{s_b, s_{ab}\} e = i \frac{d}{d\tau} \left(b + \bar{b} + 2 \beta \bar{\beta}\right), \qquad (13)$$

is *true* if and only if the CF-type restriction $b + \bar{b} + 2\beta\bar{\beta} = 0$ is invoked from *outside*. However, this restriction is a *physical* constraint because *it* is an (anti-)BRST invariant (i.e., $s_{(a)b}[b + \bar{b} + 2\beta\bar{\beta}] = 0$) quantity. It can be readily checked that

$$\{s_b, s_{ab}\} \Phi = 0,$$
$$\Phi = p_\mu, \psi_\mu, \psi_5, b, \bar{b}, \beta, \bar{\beta}, c, \bar{c}, \gamma. \qquad (14)$$

In other words, we observe that the off-shell nilpotent ($s_{(a)b}^2 = 0$) (anti-)BRST symmetry transformations ($s_{(a)b}$) are absolutely anticommuting (i.e., $\{s_b, s_{ab}\} = s_b s_{ab} + s_{ab} s_b = 0$) provided the entire theory is considered on the quantum submanifold where the CF-type restriction $b + \bar{b} + 2\beta\bar{\beta} = 0$ is satisfied. It is the existence of *this* physical restriction that leads to the existence of coupled (but equivalent) Lagrangians

$$L_b = L_f + s_b s_{ab} \left[\frac{i}{2} e^2 + \chi \psi_5 - \frac{1}{2} \bar{c} c\right],$$
$$L_{\bar{b}} = L_f - s_{ab} s_b \left[\frac{i}{2} e^2 + \chi \psi_5 - \frac{1}{2} \bar{c} c\right], \qquad (15)$$

which incorporate the gauge-fixing and Faddeev-Popov ghost terms in addition to the first-order Lagrangian (L_f) of Equation (1). In the full blaze of glory, the above Lagrangians (in terms of *all* the appropriate variables) are as follows [6, 26]:

$$L_b = L_f + b(\dot{e} + 2\bar{\beta}\beta) + b^2 - i\dot{\bar{c}}\dot{c} + \bar{\beta}^2 \beta^2 - 2e\left(\bar{\beta}\dot{\beta} + \gamma\chi\right)$$
$$+ 2i\chi(\beta\dot{\bar{c}} - \bar{\beta}\dot{c}) + m\left(\bar{\beta}\dot{\beta} - \dot{\bar{\beta}}\beta + \gamma\chi\right) + 2\gamma(\beta\bar{c} - \bar{\beta}c) - \dot{\gamma}\psi_5, \qquad (16)$$

$$L_{\bar{b}} = L_f - \bar{b}(\dot{e} - 2\bar{\beta}\beta) + \bar{b}^2 - i\dot{\bar{c}}\dot{c} + \bar{\beta}^2 \beta^2 + 2i\chi(\beta\dot{\bar{c}} - \bar{\beta}\dot{c})$$
$$+ 2e\left(\dot{\bar{\beta}}\beta - \gamma\chi\right) + m\left(\bar{\beta}\dot{\beta} - \dot{\bar{\beta}}\beta + \gamma\chi\right) + 2\gamma(\beta\bar{c} - \bar{\beta}c) - \dot{\gamma}\psi_5, \qquad (17)$$

where the subscripts b and $\bar{b}$ are appropriate because the Lagrangian L_b depends on the Nakanishi-Lautrup auxiliary variable b but the Lagrangian $L_{\bar{b}}$ contains the auxiliary variable $\bar{b}$ in its full expression. It is straightforward to check that L_b and $L_{\bar{b}}$ of our theory respect the *perfect* BRST and anti-BRST transformations because we have

$$s_b L_b = \frac{d}{d\tau} \left[\frac{\beta}{2} \left(p_\mu \psi^\mu + m \psi_5 \right) + \frac{c}{2} \left(p^2 + m^2 \right) + b \left(\dot{c} + 2\beta\chi \right) \right],$$
(18)

$$s_{ab} L_{\bar{b}} = \frac{d}{d\tau} \left[\frac{\bar{\beta}}{2} \left(p_\mu \psi^\mu + m \psi_5 \right) + \frac{\bar{c}}{2} \left(p^2 + m^2 \right) - \bar{b} \left(\dot{\bar{c}} + 2\bar{\beta}\chi \right) \right].$$
(19)

As a consequence, the action integrals $S_1 = \int_{-\infty}^{\infty} d\tau\, L_b$ and $S_2 = \int_{-\infty}^{\infty} d\tau\, L_{\bar{b}}$ remain invariant under the BRST and anti-BRST symmetry transformations (12) and (11), respectively. We define a *perfect* symmetry as the one under which the action integral remains invariant *without* any use of the CF-type restriction and/or EL-EOMs.

The BRST quantization of the massive spinning particle can be performed using the (anti-)BRST transformations (11) and (12) which correspond to the *classical* (super)gauge symmetry transformations (8). In our recent publication [26], we have discussed *all* the details of this quantization scheme. However, we have *not* touched the continuous and infinitesimal reparameterization transformations (2). We focus on the *latter classical* symmetry transformations in the *next* section for the BRST analysis as it is our modest *first* step towards our main goal to discuss the diffeomorphism invariant SUSY theories in the physical $(3+1)$-dimensional (4D) and higher dimensional $(D > 4)$ spacetime.

3. Quantum (Anti-)BRST Symmetries Corresponding to the Classical Reparameterization Symmetry

In this section, we discuss the *quantum* (anti-)BRST symmetry transformations corresponding to the *classical* infinitesimal reparameterization symmetry transformations (2). This is essential and important because we wish to perform the BRST quantization of a 1D diffeomorphism (i.e., reparameterization) invariant SUSY theory. We exploit the standard techniques and tricks of the BRST formalism to elevate the above *classical* symmetry to its counterpart *quantum* symmetries. In fact, the off-shell nilpotent $(s_{(a)b}^2 = 0)$ (anti-)BRST symmetry transformations (corresponding to the *classical* Equation (2)) are

$$
\begin{aligned}
s_{ab} \psi_\mu &= \bar{C} \dot{\psi}_\mu, \\
s_{ab} p_\mu &= \bar{C} \dot{p}_\mu, \\
s_{ab} e &= \frac{d}{d\tau} \left(\bar{C} e \right), \\
s_{ab} x_\mu &= \bar{C} \dot{x}_\mu, \\
s_{ab} \bar{C} &= \bar{C} \dot{\bar{C}}, \\
s_{ab} \chi &= \frac{d}{d\tau} \left(\bar{C} \chi \right), \\
s_{ab} \psi_5 &= \bar{C} \dot{\psi}_5, \\
s_{ab} C &= i \bar{B}, \\
s_{ab} \bar{B} &= 0, \\
s_{ab} B &= \dot{B} \bar{C} - B \dot{\bar{C}},
\end{aligned}
$$
(20)

$$
\begin{aligned}
s_b x_\mu &= C \dot{x}_\mu, \\
s_b p_\mu &= C \dot{p}_\mu, \\
s_b e &= \frac{d}{d\tau} \left(C e \right), \\
s_b \psi_\mu &= C \dot{\psi}_\mu, \\
s_b \psi_5 &= C \dot{\psi}_5, \\
s_b \chi &= \frac{d}{d\tau} \left(C \chi \right), \\
s_b \bar{C} &= i B, \\
s_b B &= 0, \\
s_b C &= C \dot{C}, \\
s_b \bar{B} &= \dot{\bar{B}} C - \bar{B} \dot{C},
\end{aligned}
$$
(21)

where B and $\bar{B}$ are the Nakanishi-Lautrup auxiliary and $(\bar{C})C$ are the (anti-)ghost variables of our theory. As far as the absolute anticommutativity property (i.e., $\{s_b, s_{ab}\} = 0$) of the above transformations is concerned, we note the following:

$$
\begin{aligned}
\{s_b, s_{ab}\} s_\mu &= i \left[B + \bar{B} + i \left(\bar{C} \dot{C} - \dot{\bar{C}} C \right) \right] \dot{s}_\mu, \\
\{s_b, s_{ab}\} \Phi &= i \frac{d}{d\tau} \left[\left\{ B + \bar{B} + i \left(\bar{C} \dot{C} - \dot{\bar{C}} C \right\} \Phi \right], \\
\{s_b, s_{ab}\} \Psi &= 0, \\
\Psi &= B, \bar{B}, C, \bar{C},
\end{aligned}
$$
(22)

where $s_\mu = x_\mu(\tau), p_\mu(\tau), \psi_\mu(\tau), \psi_5(\tau)$ and $\Phi = e(\tau), \chi(\tau)$. Thus, we note that the absolute anticommutativity property of the (anti-)BRST symmetry transformations (cf. Equations (20) and (21)) is satisfied (i.e., $\{s_b, s_{ab}\} = s_b s_{ab} + s_{ab} s_b = 0$) if and only if we *invoke* the (anti-)BRST invariant (i.e., $s_{(a)b}[B + \bar{B} + i(\bar{C}\dot{C} - \dot{\bar{C}}C)] = 0$) CF-type restriction $[B + \bar{B} + i(\bar{C}\dot{C}$

$-\dot{\bar{C}}C)=0$]. We note, therefore, that a CF-type constraint exists on our theory which is the root cause behind the absolute anticommutativity (i.e., $\{s_b, s_{ab}\} = 0$) of the (anti-)BRST symmetry transformations *and* it leads to the existence of the coupled (but equivalent) (anti-)BRST invariant Lagrangians as

$$L_B = L_f + s_b\, s_{ab}\left[\frac{i}{2}\, e^2 + \chi\, \psi_5 - \frac{1}{2}\, \bar{C}\, C\right],$$
$$L_{\bar{B}} = L_f - s_{ab}\, s_b\left[\frac{i}{2}\, e^2 + \chi\, \psi_5 - \frac{1}{2}\, \bar{C}\, C\right], \tag{23}$$

where the (anti-)BRST symmetry transformations $(s_{(a)b})$ are the *quantum* symmetries (cf. Equations (20) and (21)) corresponding to the *classical* infinitesimal reparameterization symmetry transformations (2). It will be noted that the quantities in the square brackets of (23) are the *same* as quoted in Equation (15). However, the (anti-)BRST symmetry transformations in (23) are *different* from (15) as *are* the notations for the Nakanishi-Lautrup-type auxiliary and (anti-)ghost variables (cf. Equation (11), (12), (20), and (21) for details).

The above coupled (but equivalent) Lagrangians (23) can be expressed in terms of the *auxiliary* and *basic* variables in an explicit form as

$$L_B = L_f + B\left[e\,\dot{e} - i\left(2\,\dot{\bar{C}}\,C + \bar{C}\,\dot{C}\right)\right] + \frac{B^2}{2} - i\, e^2\, \dot{\bar{C}}\,\dot{C} - i\, e\,\dot{e}\,\dot{\bar{C}}\,C - \dot{\bar{C}}\,\bar{C}\,\dot{C}\,C,$$
$$L_{\bar{B}} = L_f - \bar{B}\left[e\,\dot{e} - i\left(2\,\bar{C}\,\dot{C} + \dot{\bar{C}}\,C\right)\right] + \frac{\bar{B}^2}{2} - i\, e^2\, \dot{\bar{C}}\,\dot{C} - i\, e\,\dot{e}\,\bar{C}\,\dot{C} - \dot{\bar{C}}\,\bar{C}\,\dot{C}\,C. \tag{24}$$

We note that the pure Faddeev-Popov ghost part (i.e., $-\dot{\bar{C}}\,\bar{C}\,\dot{C}\,C$) of the above coupled (but equivalent) Lagrangians is the *same*. It can be readily checked that the EL-EOMs from Lagrangians L_B and $L_{\bar{B}}$, w.r.t. the auxiliary variables B and $\bar{B}$, lead to the following:

$$B = -e\,\dot{e} + 2\, i\,\dot{\bar{C}}\,C + i\,\bar{C}\,\dot{C},$$
$$\bar{B} = e\,\dot{e} - 2\, i\,\bar{C}\,\dot{C} - i\,\dot{\bar{C}}\,C, \tag{25}$$

which are responsible for the derivation of the CF-type restriction: $B + \bar{B} + i\,(\bar{C}\,\dot{C} - \dot{\bar{C}}\,C) = 0$. The above Lagrangians L_B and $L_{\bar{B}}$ of our theory respect the BRST and anti-BRST transformations in a precise and *perfect* manner because it is interesting to check that

$$s_b L_B = \frac{d}{d\tau}\left[C L_f + e^2\, B\,\dot{C} + e\,\dot{e}\, B\,C - i\, B\,\bar{C}\,\dot{C}\,C + B^2\, C\right], \tag{26}$$

$$s_{ab} L_{\bar{B}} = \frac{d}{d\tau}\left[\bar{C} L_f - e^2\,\bar{B}\,\dot{\bar{C}} - e\,\dot{e}\,\bar{B}\,\bar{C} - i\,\bar{B}\,\dot{\bar{C}}\,\bar{C}\,C + \bar{B}^2\,\bar{C}\right]. \tag{27}$$

As a consequence, the action integrals $S_1 = \int_{-\infty}^{\infty} d\tau\, L_B$

and $S_2 = \int_{-\infty}^{\infty} d\tau\, L_{\bar{B}}$ respect the continuous and nilpotent symmetries s_b and s_{ab} because of the Gauss's divergence theorem (where all the physical variables of our theory vanish off as $\tau \longrightarrow \pm\infty$). We can *also* apply s_b on $L_{\bar{B}}$ and s_{ab} on L_B. The ensuing results are as follows:

$$s_b L_{\bar{B}} = \frac{d}{d\tau}\Big[C L_f - e^2\left(i\,\dot{\bar{C}}\,\dot{C}\,C + \bar{B}\,\dot{C}\right) - e\,\dot{e}\left(i\,\bar{C}\,\dot{C}\,C + \bar{B}\,C\right) + i\,(2\,\bar{B} - B)\,\bar{C}\,\dot{C}\,C + \bar{B}^2\,C\Big] + \Big[B + \bar{B} + i\left(\bar{C}\,\dot{C} - \dot{\bar{C}}\,C\right)\Big] \cdot\Big[i\,\bar{C}\,\ddot{C}\,C + e\,\dot{e}\,\dot{C} - 2\,\bar{B}\,\dot{C} + 2\, i\,\dot{\bar{C}}\,\dot{C}\,C\Big] + \frac{d}{d\tau}\Big[B + \bar{B} + i\left(\bar{C}\,\dot{C} - \dot{\bar{C}}\,C\right)\Big]\left(e^2\,\dot{C} - \bar{B}\,C\right), \tag{28}$$

$$s_{ab} L_B = \frac{d}{d\tau}\Big[\bar{C} L_f + e^2\left(i\,\dot{\bar{C}}\,\bar{C}\,\dot{C} + B\,\dot{\bar{C}}\right) + e\,\dot{e}\left(i\,\dot{\bar{C}}\,\bar{C}\,C + B\,\bar{C}\right) + i\,(2\,B - \bar{B})\,\dot{\bar{C}}\,\bar{C}\,C + B^2\,\bar{C}\Big] + \Big[B + \bar{B} + i\left(\bar{C}\,\dot{C} - \dot{\bar{C}}\,C\right)\Big] \cdot\Big(i\,\ddot{\bar{C}}\,\bar{C}\,C - e\,\dot{e}\,\dot{\bar{C}} - 2\,B\,\dot{\bar{C}} + 2\, i\,\dot{\bar{C}}\,\bar{C}\,\dot{C}\Big) - \frac{d}{d\tau}\Big[B + \bar{B} + i\left(\bar{C}\,\dot{C} - \dot{\bar{C}}\,C\right)\Big]\left(e^2\,\dot{\bar{C}} + B\,\bar{C}\right), \tag{29}$$

which demonstrate that the coupled Lagrangians L_B and $L_{\bar{B}}$ of our theory are *equivalent* in the sense that *both* of them respect *both* (i.e., BRST and anti-BRST) transformations due to the validity of the *physical* CF-type restriction: $B + \bar{B} + i\,(\bar{C}\,\dot{C} - \dot{\bar{C}}\,C) = 0$.

It is very interesting to point out that the contributions of the term "$\chi\,\psi_5$" in Equation (23) turn out to be total derivatives because we observe that

$$s_b s_{ab}(\chi\,\psi_5) = \frac{d}{d\tau}\left[\left(i\,B\,\chi - \bar{C}\,\dot{C}\,\chi - \bar{C}\,C\,\dot{\chi}\right)\psi_5 - \bar{C}\,C\,\chi\,\dot{\psi}_5\right],$$
$$-s_{ab} s_b(\chi\,\psi_5) = -\frac{d}{d\tau}\left[\left(i\,\bar{B}\,\chi + \dot{\bar{C}}\,C\,\chi + \bar{C}\,C\,\dot{\chi}\right)\psi_5 + \bar{C}\,C\,\chi\,\dot{\psi}_5\right]. \tag{30}$$

As a consequence, the gauge-fixing and Faddeev-Popov ghost terms of the Lagrangians L_B and $L_{\bar{B}}$ of Equation (24) originate from the same terms as the *ones* derived in our earlier work [6] on the massless spinning relativistic particle. Thus, we note that the variable (i.e., ψ_5), corresponding to the *mass* term for a *massive* spinning relativistic particle, does *not* contribute anything *new* to the gauge-fixing and Faddeev-Popov ghost terms. In other words, the dynamics of our theory (at the BRST *quantized* level) is unaffected by the presence of the "$\chi\,\psi_5$" term. This is a *novel* observation in our theory which is radically different from our earlier work [26] where the "$\chi\,\psi_5$" term contributes to the dynamics. The observations in Equation (30) also imply that the absolute anticommutativity property $\{s_b, s_{ab}\}(\chi\,\psi_5) = i\,(d/d\tau)[B + \bar{B} + i\,(\bar{C}\,\dot{C} - \dot{\bar{C}}\,C)]$ of the (anti-)BRST symmetries $(s_{(a)b})$ is satisfied (i.e., $\{s_b, s_{ab}\} = 0$) only when the CF-type restriction is imposed from *outside*.

According to Noether's theorem, the continuous symmetry invariance of the action integrals, corresponding to the transformations (26) and (27), leads to the derivation of the conserved currents (i.e., conserved charges for our 1D system) as

$$J_B = i B \bar{C} C \dot{C} + B^2 C + B e^2 \dot{C} + B e \dot{e} C + \frac{1}{2} e C (p^2 - m^2) + i \chi C (p_\mu \psi^\mu - m \psi_5),$$
(31)

$$J_{\bar{B}} = i \bar{B} \bar{C} \dot{C} C + \bar{B}^2 \bar{C} - \bar{B} e^2 \dot{\bar{C}} - \bar{B} e \dot{e} \bar{C} + \frac{1}{2} e \bar{C} (p^2 - m^2) + i \chi \bar{C} (p_\mu \psi^\mu - m \psi_5).$$
(32)

The conservation law $(dJ_r/d\tau) = 0$ (with $r = B, \bar{B}$) can be proven by using the EL-EOMs that derived from the Lagrangians L_B and $L_{\bar{B}}$. For instance, we point out that the following EL-EOMs w.r.t. the variables $(x_\mu, p_\mu, \psi_\mu, \psi_5, \chi, e, B, \bar{B}, C, \bar{C})$, namely,

$$\dot{p}_\mu = 0,$$
$$\dot{x}_\mu = e p_\mu - i \chi \psi_\mu,$$
$$\psi_\mu = \chi p_\mu,$$
$$\dot{\psi}_5 = m \chi,$$
$$p_\mu \psi^\mu = m \psi_5,$$
$$B - i \left(2 \dot{\bar{C}} C + i e \dot{e} + \bar{C} \dot{C} \right) = 0,$$
$$e \dot{B} + i e \dot{\bar{C}} \dot{C} - i e \ddot{\bar{C}} C + \frac{1}{2} (p^2 - m^2) = 0,$$
$$i \bar{B} \bar{C} - i B \dot{\bar{C}} + i e \dot{e} \dot{C} + i e^2 \ddot{C} + \bar{C} \ddot{C} C + 2 \bar{C} \dot{C} \dot{C} = 0,$$
$$-i B \dot{C} - 2 i \dot{B} C - 3 i e \dot{e} \dot{C} - i e^2 \ddot{C} - i \dot{e}^2 C - i e \ddot{e} C + \bar{C} C \ddot{C} + 2 \dot{C} \bar{C} \dot{C} = 0,$$
(33)

are obtained from L_B. The equations of motion that are different from (33) and emerge out from $L_{\bar{B}}$ (as the EL-EOMs) are as follows:

$$\bar{B} + i \left(2 \bar{C} \dot{C} + i e \dot{e} + \dot{\bar{C}} C \right) = 0,$$
$$-e \dot{\bar{B}} + i e \dot{\bar{C}} \dot{C} - i e \bar{C} \ddot{C} + \frac{1}{2} (p^2 - m^2) = 0,$$
$$i \dot{\bar{B}} C - i \bar{B} \dot{C} - i e \dot{e} \dot{C} - i e^2 \ddot{C} + \bar{C} \ddot{C} C + 2 \dot{\bar{C}} \dot{C} C = 0,$$
$$-i \bar{B} \dot{\bar{C}} - 2 i \dot{\bar{B}} \bar{C} + 3 i e \dot{e} \dot{\bar{C}} + i e^2 \ddot{\bar{C}} + i \dot{e}^2 \bar{C} + i e \ddot{e} \bar{C} + \bar{C} \ddot{C} \bar{C} + 2 \bar{C} \dot{C} \dot{\bar{C}} = 0.$$
(34)

These conserved currents $(J_B, J_{\bar{B}})$ lead to the definition of the conserved charges Q_B and $Q_{\bar{B}}$ which are the *same* (i.e., $J_B = Q_B, J_{\bar{B}} = Q_{\bar{B}}$) as the conserved currents quoted in Equations (31) and (32). This is due to the fact that we are dealing with a 1D system.

We close this section with the following key comments. First, we observe that the off-shell nilpotent (anti-)BRST symmetry transformations (20) and (21) are absolutely anticommuting in nature provided we invoke the (anti-)BRST invariant CF-type restriction $B + \bar{B} + i (\bar{C} \dot{C} - \dot{\bar{C}} C) = 0$ from *outside*. Second, this restriction can be obtained from

the coupled (but equivalent) Lagrangians L_B and $L_{\bar{B}}$ if we use the EL-EOMs (cf. Equation (25)) w.r.t. the Nakanishi-Lautrup-type auxiliary variables B and $\bar{B}$. Third, we observe that the term $(\chi \psi_5)$ in the square bracket of Equation (23) does *not* contribute anything to the dynamics as well as to the gauge-fixing and Faddeev-Popov ghost terms. Fourth, the coupled Lagrangians L_B and $L_{\bar{B}}$ are *equivalent* in the sense that *both* of them respect *both* off-shell nilpotent (anti-)BRST symmetries on a submanifold of the *quantum* variables where the CF-type constraint $B + \bar{B} + i (\bar{C} \dot{C} - \dot{\bar{C}} C) = 0$ is satisfied. This key observation is an *alternative* proof of the existence of CF-type restriction on our theory. Finally, we observe that the absolute anticommutativity (i.e., $\{Q_B, Q_{\bar{B}}\} = 0$) of the conserved (i.e., $\dot{Q}_{(\bar{B})B} = 0$) and off-shell nilpotent (i.e., $Q_{(\bar{B})B}^2 = 0$) (anti-)BRST charges $(Q_{(\bar{B})B})$ is satisfied only due to the validity of the *existence* of the CF-type restriction (cf. Section 6).

4. Off-Shell Nilpotent Symmetries of the Target Space Variables and CF-Type Restriction: MBTSA

In this section, we derive the off-shell nilpotent (anti-)BRST symmetry transformations for the target space variables ($x_\mu, p_\mu, \psi_\mu, \psi_5$) which are *scalars* w.r.t. the 1D space of the trajectory for the massive spinning relativistic particle that is embedded in the D-dimensional target space. For this purpose, we exploit the theoretical power and potential of MBTSA (see, e.g., [22, 25]). In this context, first of all, we *promote* the 1D diffeomorphism transformation $\tau \longrightarrow \tau' = f(\tau) \equiv \tau - \varepsilon(\tau)$ (where $\varepsilon(\tau)$ is the infinitesimal transformation parameter corresponding to the 1D diffeomorphism (i.e., reparameterization) symmetry transformation) to its counterpart on the $(1, 2)$-dimensional supermanifold as

$$f(\tau) \longrightarrow f\left(\tau, \theta, \bar{\theta}\right) = \tau - \theta \bar{C} - \bar{\theta} C + \theta \bar{\theta} h(\tau), \quad (35)$$

where the general $(1, 2)$-dimensional supermanifold is parameterized by a bosonic (i.e., evolution) coordinate τ and a pair of Grassmannian variables $(\theta, \bar{\theta})$ that satisfy $\theta^2 = \bar{\theta}^2 = 0, \theta \bar{\theta} + \bar{\theta} \theta = 0$. In the above diffeomorphism transformation, the function $f(\tau)$ is any arbitrary function of the evolution parameter τ such that it is *finite* at $\tau = 0$ and vanishes off at $\tau = \pm\infty$. In other words, $f(\tau)$ is a *physically* well-defined function of τ. It is worth pointing out that the coefficients of the Grassmannian variables $(\theta, \bar{\theta})$, in Equation (35), are nothing but the fermionic (i.e., $C^2 = \bar{C}^2 = 0, C\bar{C} + \bar{C} C = 0$) (anti-)ghost variables $(\bar{C}) C$ of the (anti-)BRST symmetry transformations (20) and (21) corresponding to the infinitesimal reparameterization symmetry transformations (2). In other words, the infinitesimal reparameterization bosonic transformation parameter $\varepsilon(\tau)$ has been replaced by the fermionic (anti-)ghost variables $(\bar{C}) C$ of the (anti-)BRST symmetry transformations. This has been done purposely in view of the fact that in earlier works

(see, e.g., [10–12]), it has been established that the translational generators $(\partial_\theta, \partial_{\overline{\theta}})$, along the Grassmannian directions $(\theta, \overline{\theta})$, are intimately connected with the nilpotent (anti-)BRST transformations $s_{(a)b}$ in the *ordinary* space. In other words, we have already taken into account $s_{ab}\tau = -\overline{C}$, $s_b\tau = -C$, which are the generalization of the *classical* infinitesimal 1D diffeomorphism symmetry transformation $\delta_r\tau = -\varepsilon(\tau)$ to its *quantum* counterparts $(s_{(a)b})$ within the framework of BRST formalism. It is worthwhile to point out that the secondary variable $h(\tau)$ of the expansion (37) has to be determined from *other* consistency conditions which we elaborate in our forthcoming paragraphs.

According to the basic tenets of MBTSA, we have to *generalize all* the ordinary variables of the Lagrangians (24) onto the $(1,2)$-dimensional supermanifold as *their* counterpart supervariables where the *generalization* of the 1D diffeomorphism transformation (cf. Equation (35)) has to be incorporated (in a suitable fashion) into the expressions for the supervariables. For instance, we shall have the following generalization as far as the *generic* target space variable $s_\mu(\tau)$ (cf. Equation (22)) is concerned, namely,

$$s_\mu(\tau) \longrightarrow \tilde{S}_\mu\big[f\big(\tau,\theta,\overline{\theta}\big),\theta,\overline{\theta}\big] \equiv \tilde{S}_\mu\big[\tau - \theta\,\overline{C} - \overline{\theta}\,C + \theta\overline{\theta}\,h,\theta,\overline{\theta}\big], \tag{36}$$

where the pair of variables $(\theta,\overline{\theta})$, as pointed out earlier, is the Grassmannian variables (i.e., $\theta^2 = \overline{\theta}^2 = 0$, $\theta\overline{\theta} + \overline{\theta}\theta = 0$) of the superspace coordinates $Z^M = (\tau,\theta,\overline{\theta})$ that characterize the $(1,2)$-dimensional supermanifold on which our 1D *ordinary* theory of the reparameterization invariant massive spinning particle is considered. Now, following the techniques of MBTSA, we have the following super expansion of (36) along *all* the possible directions of the Grassmannian variables of the $(1,2)$-dimensional supermanifold, namely,

$$\tilde{S}_\mu\big[f\big(\tau,\theta,\overline{\theta}\big),\theta,\overline{\theta}\big] = S_\mu\big[f\big(\tau,\theta,\overline{\theta}\big)\big] + \theta\,\overline{R}_\mu\big[f\big(\tau,\theta,\overline{\theta}\big)\big]$$
$$+ \overline{\theta}\,R_\mu\big[f\big(\tau,\theta,\overline{\theta}\big)\big] + \theta\overline{\theta}\,P_\mu\big[f\big(\tau,\theta,\overline{\theta}\big)\big], \tag{37}$$

where on the r.h.s., we have the *secondary* supervariables which are *also* function of $f(\tau,\theta,\overline{\theta})$. As a consequence, we can have the following Taylor expansions (for those secondary variables that are present on the r.h.s.), namely,

$$S_\mu\big(\tau - \theta\,\overline{C} - \overline{\theta}\,C + \theta\overline{\theta}\,h\big) = s_\mu(\tau) - \theta\,\overline{C}\,\dot{s}_\mu(\tau) - \overline{\theta}\,C\dot{s}_\mu(\tau) + \theta\overline{\theta}\big(h\,\dot{s}_\mu - \overline{C}\,C\,\ddot{s}_\mu\big),$$

$$\theta\,\overline{R}_\mu\big(\tau - \theta\,\overline{C} - \overline{\theta}\,C + \theta\overline{\theta}\,h\big) \equiv \theta\,\overline{R}_\mu(\tau) - \theta\overline{\theta}\,\dot{\overline{R}}_\mu(\tau),$$

$$\overline{\theta}\,R_\mu\big(\tau - \theta\,\overline{C} - \overline{\theta}\,C + \theta\overline{\theta}\,h\big) \equiv \overline{\theta}\,R_\mu(\tau) + \theta\overline{\theta}\,\dot{R}_\mu(\tau),$$

$$\theta\,\overline{\theta}\,P_\mu\big(\tau - \theta\,\overline{C} - \overline{\theta}\,C + \theta\overline{\theta}\,h\big) \equiv \theta\overline{\theta}\,P_\mu(\tau). \tag{38}$$

At this juncture, we would like to lay stress on the fact that, in the super expansion (37), *all* the supervariables on the r.h.s. have to be *ordinary* variables as *all* of them are Lorentz *scalars* w.r.t. the 1D trajectory of the particle (that is embedded in the *D*-dimensional flat Minkowskian target space). It is worthwhile to point out that a pure Lorentz (bosonic or fermionic) scalar is the one which does *not* transform *at all* under any kind of *spacetime* and/or *internal* transformations. As a result, the expansion (37) can be written as

$$\tilde{S}_\mu\big[f\big(\tau,\theta,\overline{\theta}\big),\theta,\overline{\theta}\big] = s_\mu(\tau) + \theta\,\overline{R}_\mu(\tau) + \overline{\theta}\,R_\mu(\tau) + \theta\overline{\theta}\,P_\mu(\tau)$$
$$\equiv s_\mu(\tau) + \theta\big(s_{ab}\,s_\mu(\tau)\big) + \overline{\theta}\big(s_b\,s_\mu(\tau)\big) + \theta\overline{\theta}\big(s_b\,s_{ab}\,s_\mu(\tau)\big), \tag{39}$$

where $s_{(a)b}$ are the (anti-)BRST symmetry transformations (20) and (21). This is due to fact that the (anti-)BRST symmetry transformations $s_{(a)b}$ have been shown to be deeply connected with the translational generators $(\partial_\theta, \partial_{\overline{\theta}})$ along the $(\theta,\overline{\theta})$-directions of the $(1,2)$-dimensional supermanifold (see, e.g., [10–12] for details).

It is evident that we have to compute the values of R_μ, $\overline{R}_\mu$, and P_μ (in terms of the auxiliary and basic variables of the Lagrangians (24)) so that we can obtain the off-shell nilpotent (anti-)BRST symmetry transformations for the *generic* variable $s_\mu(\tau)$. At this stage, the so called "horizontality condition" (HC) comes to our help where we demand that $\tilde{S}_\mu(f(\tau,\theta,\overline{\theta}),\theta,\overline{\theta}) = s_\mu(\tau)$. This relationship can be explicitly expressed as

$$s_\mu(\tau) + \theta\big(\overline{R}_\mu - \overline{C}\,\dot{s}_\mu\big) + \overline{\theta}\big(R_\mu - C\,\dot{s}_\mu\big)$$
$$+ \theta\overline{\theta}\big[h\,\dot{s}_\mu - \overline{C}\,C\,\ddot{x}_\mu - C\,\dot{\overline{R}}_\mu + \overline{C}\,\dot{R}_\mu + P_\mu\big] \equiv s_\mu(\tau), \tag{40}$$

where we have collected all the terms from Equation (38) to express (36). Physically, the above requirement corresponds to the fact that a Lorentz-scalar does *not* transform under any kind of physically well-defined spacetime transformations. Needless to say, the relationship (40) implies that we have the following explicit relationships:

$$R_\mu = C\,\dot{s}_\mu,$$
$$\overline{R}_\mu = \overline{C}\,\dot{s}_\mu, \tag{41}$$
$$P_\mu = C\,\dot{\overline{R}}_\mu - \overline{C}\,\dot{R}_\mu + \overline{C}\,C\,\dot{s}_\mu - h\,\ddot{s}_\mu.$$

It is straightforward to note that we have already obtained $s_b\,s_\mu = C\,\dot{s}_\mu$ and $s_{ab}\,s_\mu = \overline{C}\,\dot{s}_\mu$ as is evident from Equation (39). The requirement of the absolute anticommutativity (that is *one* of the sacrosanct properties of the (anti-)BRST symmetry transformations) implies that we have the following equalities, namely,

$$s_b\,s_{ab}\,s_\mu = -s_{ab}\,s_b\,s_\mu,$$
$$\{s_b, s_{ab}\}\,s_\mu = 0. \tag{42}$$

On the other hand, the requirement of the off-shell

nilpotency (that is *another* sacrosanct property of the (anti-)BRST symmetry transformations) leads to the following:

$$s_b C = C \dot{C},$$
$$s_{ab} \bar{C} = \bar{C} \dot{\bar{C}}. \tag{43}$$

On top of the already obtained off-shell nilpotent (anti-)BRST symmetry transformations: $s_b s_\mu = C \dot{s}_\mu, s_{ab} s_\mu = \bar{C} \dot{s}_\mu,$ $s_b C = C \dot{C}, s_{ab} \bar{C} = \bar{C} \dot{\bar{C}},$ we take into account the standard (anti-)BRST symmetry transformations $s_b \bar{C} = i B$ and $s_{ab} C = i \bar{B}$ in terms of the Nakanishi-Lautrup auxiliary variables. These standard inputs lead to the determination of the l.h.s. and r.h.s. of the *first* equality in Equation (42) as [22, 25]

$$s_b s_{ab} s_\mu = \left(i B - \bar{C} \dot{C} \right) \dot{s}_\mu - \bar{C} C \ddot{s}_\mu \equiv P_\mu(\tau),$$
$$-s_{ab} s_b s_\mu = \left(-i \bar{B} - \dot{\bar{C}} C \right) \dot{s}_\mu - \bar{C} C \ddot{s}_\mu \equiv P_\mu(\tau), \tag{44}$$

where $P_\mu(\tau)$ is present in the expansion (39). A close look at Equation (44) implies that we have $B + \bar{B} + i(\bar{C} \dot{C} - \dot{\bar{C}} C) = 0$ which is nothing but the CF-type restriction. In addition, the observation of Equation (41) implies that there is an explicit expression for P_μ in terms of $h(\tau)$ (that is present in the expansion of $f(\tau, \theta, \bar{\theta})$ in Equation (35)). Plugging in the values of $R_\mu = C \dot{s}_\mu$ and $\bar{R}_\mu = \bar{C} \dot{s}_\mu$ into Equation (41) leads to [22, 25]

$$P_\mu(\tau) = -\left[\left(\dot{\bar{C}} C + \bar{C} \dot{C} + h \right) \dot{s}_\mu + \bar{C} C \ddot{s}_\mu \right]. \tag{45}$$

Comparison of (44) and (45) yields (see, e.g., [22, 25] for details)

$$h = -i B - \dot{\bar{C}} C \equiv +i \bar{B} - \bar{C} \dot{C} B + \bar{B} + i \left(\bar{C} \dot{C} + \dot{\bar{C}} C \right) = 0. \tag{46}$$

Thus, we note that the comparison of the values of $h(\tau)$ (that is determined from the comparison between Equations (44) and (45)) leads to the deduction of the (anti-)BRST invariant (i.e., $s_{(a)b}[B + \bar{B} + i(\bar{C} \dot{C} - \dot{\bar{C}} C)] = 0$) CF-type restriction $B + \bar{B} + i(\bar{C} \dot{C} - \dot{\bar{C}} C) = 0$ which plays an important role in the proof: $\{s_b, s_{ab}\} = 0$.

We wrap up this section with the following useful and important remarks. First, we have taken into account the standard choice in the BRST formalism which is $s_b \bar{C} = i B,$ $s_{ab} C = i \bar{B}$. In other words, we have made the following (anti-)chiral super expansions for the (anti-)chiral supervariables (in view of $\partial_{\bar{\theta}} \leftrightarrow s_{ab}, \partial_\theta \leftrightarrow s_b$,), namely,

$$C(\tau) \longrightarrow F^{(c)}(\tau, \theta) = C(\tau) + \theta[i \bar{B}(\tau)] \equiv C(\tau) + \theta[s_{ab} C(\tau)],$$
$$\bar{C}(\tau) \longrightarrow \bar{F}^{(ac)}(\tau, \bar{\theta}) = \bar{C}(\tau) + \bar{\theta}[i B(\tau)] \equiv \bar{C}(\tau) + \bar{\theta}[s_b \bar{C}(\tau)], \tag{47}$$

where $F^{(c)}(\tau, \theta)$ and $\bar{F}^{(ac)}(\tau, \bar{\theta})$ are the *chiral* and antichiral supervariables that have been defined on the $(1, 1)$-dimensional suitably chosen *chiral* and *antichiral* super submanifolds of the *general* $(1, 2)$-dimensional supermanifold. Second, we have seen that (cf. Equation (22)) the absolute anticommutativity property (i.e., $\{s_b, s_{ab}\} = 0$) of the (anti-)BRST transformations $(s_{(a)b})$ is satisfied if and only if the CF-type restriction (46) is satisfied. Third, we point out that the requirement of the following

$$\{s_b, s_{ab}\} C = 0,$$
$$s_b \bar{B} = \dot{\bar{B}} C - \bar{B} \dot{C},$$
$$\{s_b, s_{ab}\} \bar{C} = 0,$$
$$s_{ab} B = \dot{B} \bar{C} - B \dot{\bar{C}}, \tag{48}$$

leads to the derivation of (anti-)BRST symmetry transformations on the Nakanishi-Lautrup auxiliary variables $(B)\bar{B}$. Fourth, within the framework of MBTSA, the CF-type restriction (46) is derived from the expression for $h(\tau)$ due to the consistency condition (i.e., $s_b s_{ab} s_\mu = -s_{ab} s_b s_\mu \equiv P_\mu$). Fifth, the (anti-)BRST symmetry transformations $(s_b s_\mu = C \dot{s}_\mu, s_{ab} s_\mu = \bar{C} \dot{s}_\mu)$ on the generic variable $s_\mu \equiv x_\mu, p_\mu,$ ψ_μ, ψ_5 imply that we have already obtained the following (anti-)BRST symmetry transformations:

$$s_{ab} x_\mu = \bar{C} \dot{x}_\mu,$$
$$s_{ab} p_\mu = \bar{C} \dot{p}_\mu,$$
$$s_{ab} \psi_\mu = \bar{C} \dot{\psi}_\mu,$$
$$s_{ab} \psi_5 = \bar{C} \dot{\psi}_5,$$
$$s_b x_\mu = C \dot{x}_\mu,$$
$$s_b p_\mu = C \dot{p}_\mu,$$
$$s_b \psi_\mu = C \dot{\psi}_\mu,$$
$$s_b \psi_5 = C \dot{\psi}_5, \tag{49}$$

for the target space variables $(x_\mu, p_\mu, \psi_\mu, \psi_5)$ of our theory that are present in the first-order Lagrangian L_f (cf. Equation (1)) for the massive spinning (i.e., SUSY) relativistic particle. Finally, the explicit form of Equation (39) can be written, after the application of HC, as follows:

$$X_\mu^{(h)} \left(f(\tau, \theta, \bar{\theta}), \theta, \bar{\theta} \right) = x_\mu(\tau) + \theta(\bar{C} \dot{x}_\mu) + \bar{\theta}(C \dot{x}_\mu)$$
$$+ \theta \bar{\theta} \left[\left(i B - \bar{C} \dot{C} \right) \dot{x}_\mu - \bar{C} C \ddot{x}_\mu \right]$$
$$\equiv x_\mu(\tau) + \theta(s_{ab} x_\mu) + \bar{\theta}(s_b x_\mu) + \theta \bar{\theta}(s_b s_{ab} x_\mu),$$

$$P_\mu^{(h)} \left(f(\tau, \theta, \bar{\theta}), \theta, \bar{\theta} \right) = p_\mu(\tau) + \theta \left(\bar{C} \dot{p}_\mu \right) + \bar{\theta}(C \dot{p}_\mu)$$
$$+ \theta \bar{\theta} \left[\left(i B - \bar{C} \dot{C} \right) \dot{p}_\mu - \bar{C} C \ddot{p}_\mu \right]$$
$$\equiv p_\mu(\tau) + \theta(s_{ab} p_\mu) + \bar{\theta}(s_b p_\mu) + \theta \bar{\theta}(s_b s_{ab} p_\mu),$$

$$\Psi_\mu^{(h)}\left(f\left(\tau,\theta,\overline{\theta}\right),\theta,\overline{\theta}\right) = \psi_\mu(\tau) + \theta\left(\overline{C}\dot{\psi}_\mu\right) + \overline{\theta}\left(C\dot{\psi}_\mu\right)$$
$$+ \theta\overline{\theta}\left[\left(iB - \overline{C}\dot{C}\right)\dot{\psi}_\mu - \overline{C}C\ddot{\psi}_\mu\right]$$
$$\equiv \psi_\mu(\tau) + \theta\left(s_{ab}\psi_\mu\right) + \overline{\theta}\left(s_b\psi_\mu\right) + \theta\overline{\theta}\left(s_b s_{ab}\psi_\mu\right),$$

$$\Psi_5^{(h)}\left(f\left(\tau,\theta,\overline{\theta}\right),\theta,\overline{\theta}\right) = \psi_5(\tau) + \theta\left(\overline{C}\dot{\psi}_5\right) + \overline{\theta}(C\dot{\psi}_5)$$
$$+ \theta\overline{\theta}\left[\left(iB - \overline{C}\dot{C}\right)\dot{\psi}_5 - \overline{C}C\ddot{\psi}_5\right]$$
$$\equiv \psi_5(\tau) + \theta\left(s_{ab}\psi_5\right) + \overline{\theta}(s_b\psi_5) + \theta\overline{\theta}(s_b s_{ab}\psi_5),$$

$$(50)$$

where the superscript (h) denotes the full expansion of the supervariables after the application of HC. A straightforward comparison of (39) with (50) shows that we have already derived the (anti-)BRST symmetry transformations (49) as the coefficients of $(\theta)\overline{\theta}$ in the super expansions (50) along with $s_b s_{ab} x_\mu, s_b s_{ab} \psi_\mu, s_b s_{ab} p_\mu, s_b s_{ab} \psi_5$ which are the coefficients of $\theta\overline{\theta}$. We also note, from Equation (50), that we have a mapping: $s_b \leftrightarrow \partial_{\overline{\theta}}|_{\theta=0}, s_{ab} \leftrightarrow \partial_\theta|_{\overline{\theta}=0}$. This observation is consistent with results obtained in Refs. [10–12].

5. Off-Shell Nilpotent (Anti-)BRST Symmetries for Other Variables of Our Theory: ACSA

In this section, we exploit the basic principle behind ACSA to BRST formalism to obtain, first of all, the off-shell nilpotent BRST symmetry transformations (21) by generalizing the basic and auxiliary variables of the Lagrangian L_B (cf. Equation (24)) on the *antichiral* $(1,1)$-dimensional super submanifold (of the most *general* $(1,2)$-dimensional supermanifold) as

$$B(\tau) \longrightarrow \tilde{B}\left(\tau,\overline{\theta}\right) = B(\tau) + \overline{\theta}f_1(\tau),$$

$$e(\tau) \longrightarrow E\left(\tau,\overline{\theta}\right) = e(\tau) + \overline{\theta}f_2(\tau),$$

$$\chi(\tau) \longrightarrow K\left(\tau,\overline{\theta}\right) = \chi(\tau) + \overline{\theta}b_1(\tau),$$

$$C(\tau) \longrightarrow F\left(\tau,\overline{\theta}\right) = C(\tau) + \overline{\theta}b_2(\tau),$$

$$\overline{C}(\tau) \longrightarrow \overline{F}\left(\tau,\overline{\theta}\right) = \overline{C}(\tau) + \overline{\theta}b_3(\tau),$$

$$\overline{B}(\tau) \longrightarrow \widetilde{\overline{B}}\left(\tau,\overline{\theta}\right) = \overline{B}(\tau) + \overline{\theta}f_3(\tau),$$

$$(51)$$

where (f_1, f_2, f_3) are the fermionic and (b_1, b_2, b_3) are bosonic *secondary* variables. These *variables* have to be expressed in terms of the auxiliary and basic variables that are present in L_B. For this purpose, we exploit the BRST-invariant restrictions. It will be noted that the antichiral $(1,1)$-dimensional super submanifold is parameterized by the superspace coordinates $Z^M = (\tau,\overline{\theta})$ where τ is the *bosonic* evolution parameter and Grassmannian variable $\overline{\theta}$ is fermionic ($\overline{\theta}^2 = 0$) in nature. In addition to (51), we have the

antichiral limit (i.e., $\theta = 0$) of the expansions (50) as follows:

$$X_\mu^{(ha)}\left(\tau,\overline{\theta}\right) = x_\mu(\tau) + \overline{\theta}(C\dot{x}_\mu),$$

$$\Psi_\mu^{(ha)}\left(\tau,\overline{\theta}\right) = \psi_\mu(\tau) + \overline{\theta}\left(C\dot{\psi}_\mu\right),$$

$$P_\mu^{(ha)}\left(\tau,\overline{\theta}\right) = p_\mu(\tau) + \overline{\theta}\left(C\dot{p}_\mu\right),$$

$$\Psi_5^{(ha)}\left(\tau,\overline{\theta}\right) = \psi_5(\tau) + \overline{\theta}(C\dot{\psi}_5),$$

$$(52)$$

where the superscript (ha) denotes the antichiral limit of the super expansions (of the supervariables (cf. Equation (50))) that have been obtained after the application of HC. It is straightforward to note that the BRST invariance ($s_b B = 0$) of the variable B implies that we have the following (with $f_1(\tau) = 0$), namely,

$$\tilde{B}\left(\tau,\overline{\theta}\right) \longrightarrow \tilde{B}^{(b)}\left(\tau,\overline{\theta}\right) = B(\tau) + \overline{\theta}(0) = B(\tau) + \overline{\theta}(s_b B),$$

$$(53)$$

where the superscript (b) stands for the antichiral supervariable that has been obtained after the BRST invariant ($s_b B = 0$) restriction. In other words, we have already obtained the BRST symmetry transformation $s_b B = 0$ as the coefficient of $\overline{\theta}$ in (53) due to our knowledge of $s_b \leftrightarrow \partial_{\overline{\theta}}$ (i.e., $\partial_{\overline{\theta}}\tilde{B}^{(b)}(\tau,\overline{\theta}) = s_b B = 0$).

The off-shell nilpotency of the BRST symmetry transformations (21) ensures that we have the following BRST invariant quantities:

$$s_b\left(C\dot{x}_\mu\right) = 0,$$

$$s_b\left(C\dot{p}_\mu\right) = 0,$$

$$s_b\left(C\dot{C}\right) = 0,$$

$$s_b\left(C\dot{e} + \dot{C}e\right) = 0,$$

$$s_b\left(\dot{C}\chi + C\dot{\chi}\right) = 0,$$

$$s_b\left(C\dot{\psi}_\mu\right) = 0,$$

$$s_b(C\dot{\psi}_5) = 0,$$

$$s_b\left(\dot{\overline{B}}C - \overline{B}\dot{C}\right) = 0.$$

$$(54)$$

The above *quantum* gauge (i.e., BRST) invariant quantities must be independent of the Grassmannian variable $\overline{\theta}$ when they are generalized onto $(1,1)$-dimensional *antichiral* super submanifold (of the most *general* $(1,2)$-dimensional supermanifold)) on which our 1D *ordinary* theory has been generalized. In other words, we have the validity of the following equalities in terms of the supervariables and ordinary variables, namely,

$$F\left(\tau,\overline{\theta}\right)\dot{X}_{\mu}^{(ha)}\left(\tau,\overline{\theta}\right) = C(\tau)\dot{x}_{\mu}(\tau),$$

$$F\left(\tau,\overline{\theta}\right)\dot{F}\left(\tau,\overline{\theta}\right) = C(\tau)\dot{C}(\tau),$$

$$F\left(\tau,\overline{\theta}\right)\dot{\Psi}_{\mu}^{(ha)}\left(\tau,\overline{\theta}\right) = C(\tau)\dot{\psi}_{\mu}(\tau),$$

$$F\left(\tau,\overline{\theta}\right)\dot{\Psi}_{5}^{(ha)}\left(\tau,\overline{\theta}\right) = C(\tau)\dot{\psi}_{5}(\tau),$$

$$F\left(\tau,\overline{\theta}\right)\dot{P}_{\mu}^{(ha)}\left(\tau,\overline{\theta}\right) = C(\tau)\dot{p}_{\mu}(\tau),$$

$$\tilde{B}^{(b)}\left(\tau,\overline{\theta}\right) = B(\tau),$$

$$\dot{F}\left(\tau,\overline{\theta}\right)E\left(\tau,\overline{\theta}\right) + F\left(\tau,\overline{\theta}\right)\dot{E}\left(\tau,\overline{\theta}\right) = \dot{C}(\tau)e(\tau) + C(\tau)\dot{e}(\tau),$$

$$\dot{F}\left(\tau,\overline{\theta}\right)K\left(\tau,\overline{\theta}\right) + F\left(\tau,\overline{\theta}\right)\dot{K}\left(\tau,\overline{\theta}\right) = \dot{C}(\tau)\chi(\tau) + C(\tau)\dot{\chi}(\tau),$$

$$\dot{\tilde{\overline{B}}}\left(\tau,\overline{\theta}\right)F\left(\tau,\overline{\theta}\right) - \tilde{\overline{B}}\left(\tau,\overline{\theta}\right)\dot{F}\left(\tau,\overline{\theta}\right) = \dot{\overline{B}}(\tau)C(\tau) - \overline{B}(\tau)\dot{C}(\tau),$$

$$(55)$$

where the supervariables with superscripts (ha) and (b) have already been explained and derived in Equations (52) and (53). The substitutions of the expansions from (52) and (51) lead to the determination of the secondary variables of the *latter* equation (cf. Equation (51)) as

$$f_2(\tau) = \dot{C}\,e + C\,\dot{e},$$

$$b_1(\tau) = \dot{C}\,\chi + C\,\dot{\chi},$$

$$b_2(\tau) = C\,\dot{C}, \tag{56}$$

$$b_3(\tau) = i\,B,$$

$$f_3(\tau) = \dot{\overline{B}}\,C - \overline{B}\,\dot{C}.$$

The above relationships demonstrate that we have already obtained the secondary variables of the super expansion (51) in terms of the auxiliary and basic variables of L_B (and the Nakanishi-Lautrup auxiliary variable $\overline{B}(\tau)$ of the Lagrangian $L_{\overline{B}}$ (cf. Equation (24))).

The substitutions of the above expressions for the secondary variables into the super expansions (51) (besides Equations (52) and (53)) are as follows:

$$E^{(b)}\left(\tau,\overline{\theta}\right) = e(\tau) + \overline{\theta}\left(e\,\dot{C} + \dot{e}\,C\right) \equiv e(\tau) + \overline{\theta}\left(s_b\,e\right),$$

$$K^{(b)}\left(\tau,\overline{\theta}\right) = \chi(\tau) + \overline{\theta}\left(C\,\dot{\chi} + \dot{C}\,\chi\right) \equiv \chi(\tau) + \overline{\theta}\left(s_b\,\chi\right),$$

$$F^{(b)}\left(\tau,\overline{\theta}\right) = C(\tau) + \overline{\theta}\left(C\,\dot{C}\right) \equiv C(\tau) + \overline{\theta}\left(s_b\,C\right),$$

$$\overline{F}^{(b)}\left(\tau,\overline{\theta}\right) = \overline{C}(\tau) + \overline{\theta}\left(i\,B\right) \equiv \overline{C}(\tau) + \overline{\theta}\left(s_b\,\overline{C}\right),$$

$$\tilde{\overline{B}}^{(b)}\left(\tau,\overline{\theta}\right) = \overline{B}(\tau) + \overline{\theta}\left(\dot{\overline{B}}\,C - \overline{B}\,\dot{C}\right) \equiv \overline{B}(\tau) + \overline{\theta}\left(s_b\,\overline{B}\right),$$

$$(57)$$

where the superscript (b) on the supervariables denotes the *antichiral* supervariables that have been obtained after the imposition of the BRST (i.e., *quantum* gauge) invariant restrictions in Equation (55). It is clear from (57) that we have a mapping $\partial_{\overline{\theta}} \leftrightarrow s_b$ which demonstrates that the off-shell nilpotency ($\partial_{\overline{\theta}}^2 = 0$) of the translational generator $\partial_{\overline{\theta}}$ along $\overline{\theta}$-direction of the $(1, 1)$-dimensional *antichiral* super submanifold *and* off-shell nilpotency ($s_b^2 = 0$) of the BRST transformations (21) in the *ordinary* space are interrelated. A careful look at Equations (50) and (57) demonstrates that we have already derived the BRST symmetry transformations (21) for *all* the variables of L_B as the coefficients of $\overline{\theta}$.

Now, we dwell a bit on the derivation of the anti-BRST transformations (20) within the framework of ACSA. Towards this end in mind, we note that the following are the *chiral* (i.e., $\overline{\theta} = 0$) limit of the full super expansions in (50), namely,

$$X_{\mu}^{(hc)}(\tau,\theta) = x_{\mu}(\tau) + \theta\left(\overline{C}\,\dot{x}_{\mu}\right) \equiv x_{\mu}(\tau) + \theta\left(s_{ab}\,x_{\mu}\right),$$

$$P_{\mu}^{(hc)}(\tau,\theta) = p_{\mu}(\tau) + \theta\left(\overline{C}\,\dot{p}_{\mu}\right) \equiv p_{\mu}(\tau) + \theta\left(s_{ab}\,p_{\mu}\right),$$

$$\Psi_{\mu}^{(hc)}(\tau,\theta) = \psi_{\mu}(\tau) + \theta\left(\overline{C}\,\dot{\psi}_{\mu}\right) \equiv \psi_{\mu}(\tau) + \theta\left(s_{ab}\,\psi_{\mu}\right),$$

$$\Psi_{5}^{(hc)}(\tau,\theta) = \psi_5(\tau) + \theta\left(\overline{C}\,\dot{\psi}_5\right) \equiv \psi_5(\tau) + \theta\left(s_{ab}\,\psi_5\right),$$

$$(58)$$

where the superscript (hc) stands for the *chiral* limit of the supervariables that have been derived after the application of HC in Equation (50). The above expansions in (58) would be utilized in the anti-BRST invariant restrictions on the *chiral* supervariables which we are going to discuss as follows. It can be readily checked that we have the following interesting anti-BRST invariant quantities:

$$s_{ab}\left(\overline{C}\,\dot{x}_{\mu}\right) = 0,$$

$$s_{ab}\left(\overline{C}\,\dot{p}_{\mu}\right) = 0,$$

$$s_{ab}\left(\overline{C}\,\dot{C}\right) = 0,$$

$$s_{ab}\left(\dot{C}\,e + \overline{C}\,\dot{e}\right) = 0,$$

$$s_{ab}\left(\dot{C}\,\chi + \overline{C}\,\dot{\chi}\right) = 0, \tag{59}$$

$$s_{ab}\left(\overline{C}\,\dot{\psi}_{\mu}\right) = 0,$$

$$s_{ab}\left(\overline{C}\,\dot{\psi}_5\right) = 0,$$

$$s_{ab}\left(\dot{B}\,\overline{C} - B\,\dot{\overline{C}}\right) = 0,$$

where the anti-BRST symmetry transformations (s_{ab}) are the ones that have been listed in Equation (20). Keeping in our mind the mapping $s_{ab} \leftrightarrow \partial_{\theta}$ and the observation $s_{ab}\,\overline{B} = 0$

,we have the following:

$$\widetilde{\bar{B}}(\tau, \theta) \longrightarrow \widetilde{\bar{B}}^{(ab)}(\tau, \theta) = \bar{B}(\tau) + \theta(0) = \bar{B}(\tau) + \theta(s_{ab}\bar{B}),$$
(60)

where the superscript (ab) denotes the expansion of the supervariable that has been obtained after the application of the anti-BRST invariant restriction $\widetilde{\bar{B}}(\tau, \theta) = \bar{B}(\tau)$ that is obtained due to the anti-BRST invariance $(s_{ab}\bar{B} = 0)$. We also note that $\partial_\theta \widetilde{\bar{B}}^{(ab)}(\tau, \theta) = s_{ab}\bar{B} = 0$. Exploiting the basic principle of ACSA to BRST formalism, we obtain the following equalities in terms of *chiral* and *ordinary* variables, namely,

$$\bar{F}(\tau, \theta)\dot{X}_\mu^{(hc)}(\tau, \theta) = \bar{C}(\tau)\dot{x}_\mu(\tau),$$

$$\bar{F}(\tau, \theta)\dot{P}_\mu^{(hc)}(\tau, \theta) = \bar{C}(\tau)\dot{p}_\mu(\tau),$$

$$\bar{F}(\tau, \theta)\dot{\Psi}_\mu^{(hc)}(\tau, \theta) = \bar{C}(\tau)\dot{\psi}_\mu(\tau),$$

$$\bar{F}(\tau, \theta)\dot{\Psi}_5^{(hc)}(\tau, \theta) = \bar{C}(\tau)\dot{\psi}_5(\tau),$$

$$\bar{F}(\tau, \theta)\dot{\bar{F}}(\tau, \theta) = \bar{C}(\tau)\dot{\bar{C}}(\tau),$$

$$\widetilde{\bar{B}}^{(ab)}(\tau, \theta) = \bar{B}(\tau),$$

$$\dot{\bar{F}}(\tau, \theta)E(\tau, \theta) + \bar{F}(\tau, \theta)\dot{E}(\tau, \theta) = \dot{\bar{C}}(\tau)e(\tau) + \bar{C}(\tau)\dot{e}(\tau),$$

$$\dot{\bar{F}}(\tau, \theta)K(\tau, \theta) + \bar{F}(\tau, \theta)\dot{K}(\tau, \theta) = \dot{\bar{C}}(\tau)\chi(\tau) + \bar{C}(\tau)\dot{\chi}(\tau),$$

$$\dot{\tilde{B}}(\tau, \theta)\bar{F}(\tau, \theta) - \tilde{B}(\tau, \theta)\dot{\bar{F}}(\tau, \theta) = \dot{B}(\tau)\bar{C}(\tau) - B(\tau)\dot{\bar{C}}(\tau),$$
(61)

where the *chiral* supervariables with superscripts (hc) and (ab) have been discussed and explained in Equations (58) and (60). It is worth pointing out that the equalities in (61) are nothing but the generalization of our observations in (59) where the anti-BRST invariant quantities have been obtained (because of the off-shell nilpotency of the anti-BRST symmetry transformations (20)). At this stage, it is crucial to point out that, besides our *chiral* supervariables in (58) and (60), we have the following generalizations:

$$B(\tau) \longrightarrow \tilde{B}(\tau, \theta) = B(\tau) + \theta\bar{f}_1(\tau),$$

$$e(\tau) \longrightarrow E(\tau, \theta) = e(\tau) + \theta\bar{f}_2(\tau),$$

$$\chi(\tau) \longrightarrow K(\tau, \theta) = \chi(\tau) + \theta\bar{b}_1(\tau),$$
(62)

$$C(\tau) \longrightarrow F(\tau, \theta) = C(\tau) + \theta\bar{b}_2(\tau),$$

$$\bar{C}(\tau) \longrightarrow \bar{F}(\tau, \theta) = \bar{C}(\tau) + \theta\bar{b}_3(\tau).$$

The above *chiral* supervariables are defined, and their expansions have been carried out on a $(1, 1)$-dimensional super submanifold that is characterized by the superspace coordinates $Z^M = (\tau, \theta)$ where the Grassmannian coordinate θ is fermionic $(\theta^2 = 0)$ in nature. It is straightforward to

draw the conclusion that the secondary variables $(\bar{b}_1, \bar{b}_2, \bar{b}_3)$ and $(\bar{f}_1, \bar{f}_2, \bar{f}_3)$, on the r.h.s. of Equation (62), are bosonic and fermionic sets, respectively.

The stage is now set to utilize the equalities (61) where we have to plug in the chiral super expansions (58) as well as the chiral generalizations (62). This exercise leads to the following relationships between the secondary variables of (62) and the basic as well as auxiliary variables of Lagrangian $L_{\bar{B}}$ (and the Nakanishi-Lautrup auxiliary variable B of the perfectly BRST invariant Lagrangian L_B), namely,

$$\bar{f}_1 = \dot{B}\bar{C} - B\dot{\bar{C}},$$

$$\bar{f}_2 = \bar{C}\dot{e} + \dot{\bar{C}}e,$$

$$\bar{b}_1 = \bar{C}\dot{\chi} + \dot{\bar{C}}\chi,$$
(63)

$$\bar{b}_2 = i\bar{B},$$

$$\bar{b}_3 = \bar{C}\dot{\bar{C}}.$$

In other words, we have already determined the secondary variables (i.e., the coefficients of θ) of the super expansions (62). The substitutions of (63) into the *chiral* super expansion (62) on the $(1, 1)$-dimensional *chiral* super submanifold leads to the following:

$$\tilde{B}^{(ab)}(\tau, \theta) = B(\tau) + \theta\left(\dot{B}\bar{C} - B\dot{\bar{C}}\right) \equiv B(\tau) + \theta(s_{ab}B),$$

$$E^{(ab)}(\tau, \theta) = e(\tau) + \theta\left(e\dot{\bar{C}} + \dot{e}\bar{C}\right) \equiv e(\tau) + \theta(s_{ab}e),$$

$$K^{(ab)}(\tau, \theta) = \chi(\tau) + \theta\left(\dot{\bar{C}}\chi + \bar{C}\dot{\chi}\right) \equiv \chi(\tau) + \theta(s_{ab}\chi),$$

$$F^{(ab)}(\tau, \theta) = C(\tau) + \theta\left(i\bar{B}\right) \equiv C(\tau) + \theta(s_{ab}C),$$

$$\bar{F}^{(ab)}(\tau, \theta) = \bar{C}(\tau) + \theta\left(\bar{C}\dot{\bar{C}}\right) \equiv \bar{C}(\tau) + \theta(s_{ab}\bar{C}),$$
(64)

where the superscript (ab) denotes the *chiral* supervariables that have been obtained after the application of the anti-BRST (i.e., *quantum* gauge) invariant restrictions in Equation (61). It is evident, from Equation (64), that we have already obtained the anti-BRST symmetry transformations (cf. Equation (20)) of the variables $(B, e, \chi, C, \bar{C})$ as the coefficients of θ in the *chiral* super expansions (62). We also observe, in the above chiral super expansions, that there is a mapping $\partial_\theta \leftrightarrow s_{ab}$ which agrees with the result of Refs. [10–12]. For the sake of completeness, we perform the step-by-step computations of the mathematical relationships obtained in Equation (63) in our Appendix A.

6. Symmetry Invariance of the Lagrangians: ACSA

In this section, we use the results of the previous section to capture the (anti-)BRST invariance of the Lagrangians (cf. Equations (26)–(29)) within the framework of ACSA. To accomplish this goal, first of all, we generalize the

Lagrangians (24) to their counterparts (anti-)chiral *super* Lagrangians as follows:

$$
\begin{aligned}
L_B \longrightarrow \tilde{L}_B^{(ac)}\left(\tau,\overline{\theta}\right) &= \tilde{L}_f^{(ac)}\left(\tau,\overline{\theta}\right) + \tilde{B}^{(b)}\left(\tau,\overline{\theta}\right) \\
&\cdot \left[E^{(b)}\left(\tau,\overline{\theta}\right)\dot{E}^{(b)}\left(\tau,\overline{\theta}\right) - i\left\{2\,\dot{\overline{F}}^{(b)}\left(\tau,\overline{\theta}\right)F^{(b)}\left(\tau,\overline{\theta}\right) + \overline{F}^{(b)}\left(\tau,\overline{\theta}\right)\dot{F}^{(b)}\left(\tau,\overline{\theta}\right)\right\}\right] \\
&+ \frac{1}{2}\tilde{B}^{(b)}\left(\tau,\overline{\theta}\right)\tilde{B}^{(b)}\left(\tau,\overline{\theta}\right) - iE^{(b)}\left(\tau,\overline{\theta}\right)E^{(b)}\left(\tau,\overline{\theta}\right)\dot{\overline{F}}^{(b)}\left(\tau,\overline{\theta}\right)\dot{F}^{(b)}\left(\tau,\overline{\theta}\right) \\
&- iE^{(b)}\left(\tau,\overline{\theta}\right)\dot{E}^{(b)}\left(\tau,\overline{\theta}\right)\dot{\overline{F}}^{(b)}\left(\tau,\overline{\theta}\right)F^{(b)}\left(\tau,\overline{\theta}\right) \\
&- \dot{\overline{F}}^{(b)}\left(\tau,\overline{\theta}\right)\overline{F}^{(b)}\left(\tau,\overline{\theta}\right)\dot{F}^{(b)}\left(\tau,\overline{\theta}\right)F^{(b)}\left(\tau,\overline{\theta}\right),
\end{aligned}
\tag{65}
$$

$$
\begin{aligned}
L_{\bar{B}} \longrightarrow \tilde{L}_{\bar{B}}^{(c)}(\tau,\theta) &= \tilde{L}_f^{(c)}(\tau,\theta) - \tilde{\tilde{\overline{B}}}^{(ab)}(\tau,\theta) \\
&\cdot \left[E^{(ab)}(\tau,\theta)\dot{E}^{(ab)}(\tau,\theta) - i\left\{2\,\overline{F}^{(ab)}(\tau,\theta)\dot{F}^{(ab)}(\tau,\theta) + \dot{\overline{F}}^{(ab)}(\tau,\theta)F^{(ab)}(\tau,\theta)\right\}\right] \\
&+ \frac{1}{2}\tilde{\tilde{\overline{B}}}^{(ab)}(\tau,\theta)\tilde{\tilde{\overline{B}}}^{(ab)}(\tau,\theta) - iE^{(ab)}(\tau,\theta)E^{(ab)}(\tau,\theta)\dot{\overline{F}}^{(ab)}(\tau,\theta)\dot{F}^{(ab)}(\tau,\theta) \\
&- iE^{(ab)}(\tau,\theta)\dot{E}^{(ab)}(\tau,\theta)\overline{F}^{(ab)}(\tau,\theta)\dot{F}^{(ab)}(\tau,\theta) \\
&- \dot{\overline{F}}^{(ab)}(\tau,\theta)\overline{F}^{(ab)}(\tau,\theta)\dot{F}^{(ab)}(\tau,\theta)F^{(ab)}(\tau,\theta),
\end{aligned}
\tag{66}
$$

where the superscripts (ac) and (c), on the r.h.s., denote the *antichiral* and *chiral* versions of the first-order Lagrangian L_f (cf. Equation (1)). In other words, we have the following:

$$
\begin{aligned}
L_f \longrightarrow \tilde{L}_f^{(ac)}\left(\tau,\overline{\theta}\right) &= P_\mu^{(b)}\left(\tau,\overline{\theta}\right)\dot{X}^{\mu(b)}\left(\tau,\overline{\theta}\right) - \frac{1}{2}E^{(b)}\left(\tau,\overline{\theta}\right) \\
&\cdot \left[P_\mu^{(b)}\left(\tau,\overline{\theta}\right)P^{\mu(b)}\left(\tau,\overline{\theta}\right) - m^2\right] \\
&+ \frac{i}{2}\left[\Psi_\mu^{(b)}\left(\tau,\overline{\theta}\right)\dot{\Psi}^{\mu(b)}\left(\tau,\overline{\theta}\right) - \Psi_5^{(b)}\left(\tau,\overline{\theta}\right)\dot{\Psi}_5^{(b)}\left(\tau,\overline{\theta}\right)\right] \\
&+ i\tilde{\chi}^{(b)}\left(\tau,\overline{\theta}\right)\left[P_\mu^{(b)}\left(\tau,\overline{\theta}\right)\Psi^{\mu(b)}\left(\tau,\overline{\theta}\right) - m\,\Psi_5^{(b)}\left(\tau,\overline{\theta}\right)\right],
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
L_f \longrightarrow \tilde{L}_f^{(c)}(\tau,\theta) &= P_\mu^{(ab)}(\tau,\theta)\dot{X}^{\mu(ab)}(\tau,\theta) - \frac{1}{2}E^{(ab)}(\tau,\theta) \\
&\cdot \left[P_\mu^{(ab)}(\tau,\theta)P^{\mu(ab)}(\tau,\theta) - m^2\right] \\
&+ \frac{i}{2}\left[\Psi_\mu^{(ab)}(\tau,\theta)\dot{\Psi}^{\mu(ab)}(\tau,\theta) - \Psi_5^{(ab)}(\tau,\theta)\dot{\Psi}_5^{(ab)}(\tau,\theta)\right] \\
&+ i\tilde{\chi}^{(ab)}(\tau,\theta)\left[P_\mu^{(ab)}(\tau,\theta)\Psi^{\mu(ab)}(\tau,\theta) - m\,\Psi_5^{(ab)}(\tau,\theta)\right],
\end{aligned}
\tag{68}
$$

where the superscripts (b) and (ab) on the supervariables of $\tilde{L}_f^{(ac)}$ and $\tilde{L}_f^{(c)}$ have been already explained in the previous section. The superscripts (ac) and (c) on the super Lagrangians, on the l.h.s. of Equations (65) and (66), denote the *antichiral* and *chiral* generalizations of the *ordinary* Lagrangians (24). Keeping in our mind the mappings: $s_b \leftrightarrow \partial_{\overline{\theta}}, s_{ab} \leftrightarrow \partial_\theta$, we observe the (anti-)BRST invariance of the first-order Lagrangian in the language of ACSA to BRST

formalism as follows:

$$
\begin{aligned}
\frac{\partial}{\partial\overline{\theta}}\,\tilde{L}_f^{(ac)} &= \frac{d}{d\tau}\left[C\,L_f\right] \equiv s_b\,L_f, \\
\frac{\partial}{\partial\theta}\,\tilde{L}_f^{(c)} &= \frac{d}{d\tau}\left[\overline{C}\,L_f\right] \equiv s_{ab}\,L_f.
\end{aligned}
\tag{69}
$$

In other words, we have accomplished the objective of establishing a precise connection between the (anti-)BRST invariance of L_f in the *ordinary* space (cf. Equations (3), (27), and (26)) and *superspace* within the purview of ACSA. The results in (69) will be useful in the proof of the (anti-)BRST invariance (cf. Equations (27) and (26)) of the coupled (but equivalent) Lagrangians L_B and $L_{\bar{B}}$. Geometrically, it is clear from our observation in (69) that super Lagrangians $\tilde{L}_f^{(ac,c)}$ are the unique sum of (anti-)chiral supervariables (obtained after the (anti-)BRST invariant restrictions) such that their translations along $(\theta,\overline{\theta})$-directions in the *superspace* produce the *total* derivatives in the *ordinary* space.

We now focus on the BRST and anti-BRST invariance of L_B and $L_{\bar{B}}$ within the purview of ACSA. In the explicit expressions of (65) and (66), we substitute the super expansions of (52), (57), (58), and (64) and apply the derivatives $(\partial_{\overline{\theta}}, \partial_\theta)$ on them due to the mappings: $\partial_{\overline{\theta}} \leftrightarrow s_b, \partial_\theta \leftrightarrow s_{ab}$. It is straightforward to check that we have the following explicit relationships between the invariances in the *superspace* and *ordinary* space:

$$
\begin{aligned}
\frac{\partial}{\partial\overline{\theta}}\,\tilde{L}_B^{(ac)} &= \frac{d}{d\tau}\left[C\,L_f + e^2\,B\,\dot{C} + e\,\dot{e}\,B\,C - i\,B\,\overline{C}\,\dot{C}\,C + B^2\,C\right] = s_b\,L_B, \\
\frac{\partial}{\partial\theta}\,\tilde{L}_{\bar{B}}^{(c)} &= \frac{d}{d\tau}\left[\overline{C}\,L_f - e^2\,\overline{B}\,\dot{\overline{C}} - e\,\dot{e}\,\overline{B}\,\overline{C} - i\,\overline{B}\,\dot{\overline{C}}\,\overline{C}\,C + \overline{B}^2\,\overline{C}\right] = s_{ab}\,L_{\bar{B}}.
\end{aligned}
\tag{70}
$$

We would like to emphasize that the super Lagrangian $\tilde{L}_B^{(ac)}$ is a unique sum of antichiral supervariables (derived after the applications of the BRST invariant restrictions) such that its translation along $\overline{\theta}$-direction of the $(1,1)$-dimensional *antichiral* super submanifold leads to a total derivative in the *ordinary* space. The *latter* is nothing but the BRST invariance of the *ordinary* Lagrangian L_B (cf. Equation (26)). In exactly similar fashion, we can provide a geometrical interpretation for the anti-BRST invariance of $L_{\bar{B}}$ (cf. Equation (27)) in the terminology of the superspace translational generator (∂_θ) along the θ-direction of the suitably chosen *chiral* $(1,1)$-dimensional super submanifold.

At this juncture, we concentrate on the deduction of the CF-type restriction $[B + \overline{B} + i\,(\overline{C}\,\dot{C} - \dot{\overline{C}}\,C) = 0]$ in the proof of the *equivalence* between the Lagrangians L_B and $L_{\bar{B}}$ (cf. Equation (24)) within the ambit of ACSA to BRST formalism. In other words, we capture the transformations $s_{ab}\,L_B$ (cf. Equation (29)) and $s_b\,L_{\bar{B}}$ (cf. Equation (28)) in the terminology of the ACSA. Towards this central goal in our mind, first of all, we generalize the *ordinary* Lagrangian L_B to its

counterpart *chiral* super Lagrangian as

$$
\begin{aligned}
L_B \longrightarrow \tilde{L}_B^{(c)}(\tau,\theta) &= \tilde{L}_f^{(c)}(\tau,\theta) + \bar{B}^{(ab)}(\tau,\theta) \\
&\quad \cdot \left[E^{(ab)}(\tau,\theta)\dot{E}^{(ab)}(\tau,\theta) - i\left\{ 2\,\dot{\bar{F}}^{(ab)}(\tau,\theta)F^{(ab)}(\tau,\theta) + \bar{F}^{(ab)}(\tau,\theta)\dot{F}^{(ab)}(\tau,\theta) \right\} \right] \\
&\quad + \frac{1}{2}\,\bar{B}^{(ab)}(\tau,\theta)\,\bar{B}^{(ab)}(\tau,\theta) - i\,E^{(ab)}(\tau,\theta)\,E^{(ab)}(\tau,\theta)\,\dot{\bar{F}}^{(ab)}(\tau,\theta)\,\dot{F}^{(ab)}(\tau,\theta) \\
&\quad - i\,E^{(ab)}(\tau,\theta)\,\dot{E}^{(ab)}(\tau,\theta)\,\dot{\bar{F}}^{(ab)}(\tau,\theta)\,F^{(ab)}(\tau,\theta) \\
&\quad - \dot{\bar{F}}^{(ab)}(\tau,\theta)\,\bar{F}^{(ab)}(\tau,\theta)\,\dot{F}^{(ab)}(\tau,\theta)\,F^{(ab)}(\tau,\theta),
\end{aligned}
\tag{71}
$$

where the superscripts (c) and (ab) have already been explained in our earlier discussions. It is very interesting to observe that we have the following:

$$
\begin{aligned}
\frac{\partial}{\partial\theta}\tilde{L}_B^{(c)}(\tau,\theta) &= \frac{d}{d\tau}\Big[\bar{C}L_f + e^2\left(i\,\dot{\bar{C}}\,\bar{C}\,\dot{C} + B\,\dot{\bar{C}} \right) + e\,\dot{e}\left(i\,\dot{\bar{C}}\,\bar{C}\,C + B\,\bar{C} \right) \\
&\quad + i\,(2B - \bar{B})\dot{\bar{C}}\,\bar{C}\,C + B^2\,\bar{C} \Big] + \left[B + \bar{B} + i\left(\bar{C}\dot{C} - \dot{\bar{C}}\,C \right) \right] \\
&\quad \cdot \left(i\,\ddot{\bar{C}}\,\bar{C}\,C - e\,\dot{e}\,\dot{\bar{C}} - 2\,B\,\dot{\bar{C}} + 2\,i\,\dot{\bar{C}}\,\bar{C}\,\dot{C} \right) \\
&\quad - \frac{d}{d\tau}\left[B + \bar{B} + i\left(\bar{C}\dot{C} - \dot{\bar{C}}\,C \right) \right]\left(e^2\,\dot{C} + B\,\bar{C} \right) \equiv s_{ab} L_B.
\end{aligned}
\tag{72}
$$

The above equation shows that the Lagrangian L_B respects the anti-BRST symmetry transformations (20) *only* when the CF-type restriction is invoked from *outside*. In a subtle manner, we have obtained the CF-type restriction $B + \bar{B} + i\,(\bar{C}\dot{C} - \dot{\bar{C}}\,C) = 0$ within the ambit of ACSA while proving the anti-BRST invariance of the Lagrangian L_B. We now demonstrate the BRST invariance of the Lagrangian $L_{\bar{B}}$ *and* existence of the CF-type restriction $B + \bar{B} + i\,(\bar{C}\dot{C} - \dot{\bar{C}}\,C) = 0$ within the purview of theoretical tricks and techniques of ACSA to BRST formalism. Towards these aims in our mind, we generalize the Lagrangian $L_{\bar{B}}$ onto $(1, 1)$-dimensional *antichiral* super submanifold as follows:

$$
\begin{aligned}
L_{\bar{B}} \longrightarrow \tilde{L}_{\bar{B}}^{(ac)}\left(\tau,\overline{\theta}\right) &= \tilde{L}_f^{(ac)}\left(\tau,\overline{\theta}\right) - \widetilde{\bar{B}}^{(b)}\left(\tau,\overline{\theta}\right) \\
&\quad \cdot \left[E^{(b)}\left(\tau,\overline{\theta}\right)\dot{E}^{(b)}\left(\tau,\overline{\theta}\right) - i\left\{ 2\,\bar{F}^{(b)}\left(\tau,\overline{\theta}\right)\dot{F}^{(b)}\left(\tau,\overline{\theta}\right) + \dot{\bar{F}}^{(b)}\left(\tau,\overline{\theta}\right)F^{(b)}\left(\tau,\overline{\theta}\right) \right\} \right] \\
&\quad + \frac{1}{2}\widetilde{\bar{B}}^{(b)}\left(\tau,\overline{\theta}\right)\widetilde{\bar{B}}^{(b)}\left(\tau,\overline{\theta}\right) - i\,E^{(b)}\left(\tau,\overline{\theta}\right)E^{(b)}\left(\tau,\overline{\theta}\right)\dot{\bar{F}}^{(b)}\left(\tau,\overline{\theta}\right)\dot{F}^{(b)}\left(\tau,\overline{\theta}\right) \\
&\quad - i\,E^{(b)}\left(\tau,\overline{\theta}\right)\dot{E}^{(b)}\left(\tau,\overline{\theta}\right)\bar{F}^{(b)}\left(\tau,\overline{\theta}\right)\dot{F}^{(b)}\left(\tau,\overline{\theta}\right) \\
&\quad - \dot{\bar{F}}^{(b)}\left(\tau,\overline{\theta}\right)\bar{F}^{(b)}\left(\tau,\overline{\theta}\right)\dot{F}^{(b)}\left(\tau,\overline{\theta}\right)F^{(b)}\left(\tau,\overline{\theta}\right),
\end{aligned}
\tag{73}
$$

where the superscripts (ac) and (b) have been already explained in our earlier discussions. Keeping in our mind the mapping: $\partial_{\overline{\theta}} \leftrightarrow s_b$, we observe the following interesting relationship between $s_b L_{\bar{B}}$ and its counterpart in superspace, namely,

$$
\begin{aligned}
\frac{\partial}{\partial\overline{\theta}}\tilde{L}_{\bar{B}}^{(ac)}\left(\tau,\overline{\theta}\right) &= \frac{d}{d\tau}\Big[CL_f - e^2\left(i\,\dot{\bar{C}}\,\bar{C}\,C + \bar{B}\,\dot{C} \right) - e\,\dot{e}\left(i\,\bar{C}\,\dot{C}\,C + \bar{B}\,C \right) \\
&\quad + i\,(2\bar{B} - B)\,\bar{C}\,\dot{C}\,C + \bar{B}^2\,C \Big] + \left[B + \bar{B} + i\left(\bar{C}\dot{C} - \dot{\bar{C}}\,C \right) \right] \\
&\quad \cdot \left[i\,\bar{C}\,\ddot{C}\,C + e\,\dot{e}\,\dot{C} - 2\,\bar{B}\,\dot{C} + 2\,i\,\dot{\bar{C}}\,\dot{C}\,C \right] \\
&\quad + \frac{d}{d\tau}\left[B + \bar{B} + i\left(\bar{C}\dot{C} - \dot{\bar{C}}\,C \right) \right]\left(e^2\,\dot{C} - \bar{B}\,C \right) \equiv s_b L_{\bar{B}},
\end{aligned}
\tag{74}
$$

where the r.h.s. is nothing but the operation of s_b on the Lagrangian $L_{\bar{B}}$ (cf. Equation (28)). In other words, we have established an intimate relationship between the BRST symmetry transformation on $L_{\bar{B}}$ *and* operation of the translational generator $\partial_{\overline{\theta}}$ on the *antichiral* super Lagrangian $\tilde{L}_{\bar{B}}^{(ac)}(\tau,\overline{\theta})$ (defined on the $(1, 1)$-dimensional (anti-)chiral super submanifold). A careful and close look on the r.h.s. of (74) demonstrates that we have derived the CF-type restriction $B + \bar{B} + i\,(\bar{C}\dot{C} - \dot{\bar{C}}\,C) = 0$ while proving the BRST invariance of $L_{\bar{B}}$ within the purview of ACSA.

We end this section with the following concluding remarks. First, we have captured the (anti-)BRST invariance (cf. Equation (69)) of the first-order Lagrangian L_f within the ambit of ACSA. Second, we have been able to express the (anti-)BRST invariance (cf. Equations (26) and (27)) of the Lagrangians L_B and $L_{\bar{B}}$ of our theory within the purview of ACSA (cf. Equation (70)). Third, we have been able to demonstrate that our observations in Equations (28) and (29) can *also* be expressed in superspace (cf. Equations (72) and (74)) within the ambit of ACSA. Finally, we have derived the CF-type restriction $B + \bar{B} + i\,(\bar{C}\dot{C} - \dot{\bar{C}}\,C) = 0$, in a subtle manner, by expressing the transformations $s_b L_{\bar{B}}$ and $s_{ab} L_B$ in the language of ACSA (cf. Equations (72) and (74) for details). In other words, in the ordinary space, whatever we have seen in the proof of the absolute anticommutativity of (anti-)BRST symmetry transformations (cf. Equation (22)), the *same* restriction appears when we discuss $s_b L_{\bar{B}}$ and $s_{ab} L_B$ in the *superspace* by using the formal theoretical techniques of ACSA.

7. Off-Shell Nilpotency and Absolute Anticommutativity of the Conserved (Anti-)BRST Charges

In this section, we prove the off-shell nilpotency $[Q_{(\bar{B})B}^2 = 0]$ and absolute anticommutativity $[\{Q_B, Q_{\bar{B}}\} = 0]$ of the (anti-)BRST charges $Q_{(\bar{B})B}$ which have already been derived in our Section 3 where $J_B = Q_B$ and $J_{\bar{B}} = Q_{\bar{B}}$ (cf. Equations (31) and (32)). In Section 7.1, we discuss the above properties of the (anti-)BRST charges $[Q_{(\bar{B})B}]$ in the *ordinary* space. Our Section 7.2 contains the theoretical material related with the techniques of capturing the nilpotency and anticommutativity properties of the above charges within the ambit of ACSA. It is quite interesting to state that the CF-type restriction appears when we prove the absolute anticommutativity property of the charges in the *ordinary* space as well as in the *superspace* (within the purview of ACSA). We describe

explicitly the computation of $s_b L_{\bar{B}}$ in our Appendix B where the algebra is a bit more involved, and ultimately, we demonstrate the existence of the CF-type restriction in the *ordinary* space (cf. Equation (28)) which has *also* appeared in Equation (74) within the ambit of ACSA.

7.1. Off-Shell Nilpotency and Absolute Anticommutativity Properties: Ordinary Space. In this subsection, we primarily exploit the theoretical potential of the well-known relationship between the continuous symmetry transformations and their generators. In other words, we can prove the off-shell nilpotency $(Q_B^2 = Q_{\bar{B}}^2 = 0)$ of the (anti-)BRST charges $Q_{(\bar{B})B}$ in the *ordinary* space by using the standard relationship between the infinitesimal continuous (anti-)BRST transformations $(s_{(a)b})$ and their generators $(Q_{(\bar{B})B})$ as

$$s_b \, Q_B = -i\{Q_B, Q_B\} = 0,$$
$$Q_B^2 = 0,$$
$$s_{ab} \, Q_{\bar{B}} = -i\{Q_{\bar{B}}, Q_{\bar{B}}\} = 0, \tag{75}$$
$$Q_{\bar{B}}^2 = 0.$$

The above proofs of the off-shell nilpotency of the conserved charges are nothing but the reflection of the off-shell nilpotency $(s_{(a)b}^2 = 0)$ of the (anti-)BRST symmetry transformations (20) and (21) in the *ordinary* space. It would be worthwhile to point out the fact that, in the computation of the l.h.s. of (75), we have *directly* applied the (anti-)BRST symmetry transformations (20) and (21) on the appropriate form of the conserved (i.e., $\dot{Q}_{(\bar{B})B} = 0$) (anti-)BRST charges (cf. Equations (31) and (32)). In other words, the straightforward application of s_b on Q_B gives us a *zero* result. The same is the situation (i.e., $s_{ab} Q_{\bar{B}} = 0$) when we apply the anti-BRST symmetry transformations s_{ab} on the anti-BRST charge $Q_{\bar{B}}$. Hence, the Noether conserved charges (cf. Equations (31) and (32)) are off-shell nilpotent of order two (i.e., $Q_{(\bar{B})B}^2 = 0$) in the *ordinary* space due to the key relationship that is given in (75).

The above explicit proof of the off-shell nilpotency of the (anti-)BRST charges ensures that they should be able to be written as *an exact* quantity w.r.t. the off-shell nilpotent [$s_{(a)b}^2 = 0$] (anti-)BRST symmetry transformations $[s_{(a)b}]$. Towards this goal in mind, we use the following EL-EOMs (derived from the Lagrangians L_B and $L_{\bar{B}}$), namely,

$$p_\mu \psi^\mu = m \psi_5,$$
$$B = -e \dot{e} + 2 i \dot{\bar{C}} C + i \bar{C} \dot{C},$$
$$e \dot{B} + i e \dot{\bar{C}} \dot{C} - i e \ddot{\bar{C}} C + \frac{1}{2}(p^2 - m^2) = 0,$$
$$B \dot{C} + 2 \dot{B} C + 3 e \dot{e} \dot{C} + e^2 \ddot{C} + \dot{e}^2 C + e \ddot{e} C - i \bar{C} \ddot{C} C - 2 i \dot{\bar{C}} \dot{C} C = 0,$$
$$\bar{B} = e \dot{e} - 2 i \bar{C} \dot{C} - i \dot{\bar{C}} C,$$
$$e \dot{\bar{B}} - i e \dot{\bar{C}} \dot{C} + i e \bar{C} \ddot{C} - \frac{1}{2}(p^2 - m^2) = 0,$$
$$\bar{B} \dot{\bar{C}} + 2 \dot{\bar{B}} \bar{C} - 3 e \dot{e} \dot{\bar{C}} - e^2 \ddot{\bar{C}} - \dot{e}^2 \bar{C} - e \ddot{e} \bar{C} - i \ddot{\bar{C}} \bar{C} C - 2 i \dot{\bar{C}} \bar{C} \dot{C} = 0, \tag{76}$$

to get rid of the constraints $(p^2 - m^2) \approx 0$ and $(p_\mu \psi^\mu - m \psi_5) \approx 0$ from the expressions for the (anti-)BRST charges $Q_{(\bar{B})B}$ (cf. Equations (31) and (32)) to recast them as

$$Q_{\bar{B}}^{(1)} = e^2 \left[\dot{B} \bar{C} - \bar{B} \dot{C} + i \left(\ddot{\bar{C}} \bar{C} C + \dot{\bar{C}} \bar{C} \dot{C} \right) \right] + 2 i e \dot{e} \dot{\bar{C}} \bar{C} C,$$
$$Q_B^{(1)} = e^2 \left[B \dot{C} - \dot{B} C - i \left(\bar{C} \ddot{C} C + \dot{\bar{C}} \dot{C} C \right) \right] - 2 i e \dot{e} \bar{C} \dot{C} C. \tag{77}$$

We have discussed different forms of the (anti-)BRST charges in our Appendix C where the emphasis is laid on the derivation of the expressions for the (anti)BRST charges (cf. Equation (77)). The above expressions of the conserved (anti-)BRST charges can be mathematically expressed in the following *exact* forms w.r.t. the off-shell nilpotent [$s_{(a)b}^2 = 0$] (anti-)BRST symmetry transformations [$s_{(a)b}$], namely,

$$Q_{\bar{B}}^{(1)} = s_{ab} \left[i e^2 \left(\bar{C} \dot{C} - \dot{\bar{C}} C \right) \right],$$
$$Q_B^{(1)} = s_b \left[i e^2 \left(\dot{\bar{C}} C - \bar{C} \dot{C} \right) \right]. \tag{78}$$

Now it is straightforward to note that $s_{ab} Q_{\bar{B}}^{(1)} = 0$ and $s_b Q_B^{(1)} = 0$ due to the off-shell nilpotency [$s_{(a)b}^2 = 0$] of the (anti-)BRST symmetry transformations [$s_{(a)b}$]. Thus, we conclude, from our observations in Equation (78), that the nilpotency of the (anti-)BRST transformations [$s_{(a)b}$] is deeply connected with the nilpotency of their generators (anti-)BRST charges [$Q_{(\bar{B})B}$] which becomes completely transparent from the *direct* observations of the following computations:

$$\left[Q_{\bar{B}}^{(1)} \right]^2 = 0 \Leftrightarrow s_{ab} Q_{\bar{B}}^{(1)} = -i \left\{ Q_{\bar{B}}^{(1)}, Q_{\bar{B}}^{(1)} \right\} = 0 \Leftrightarrow s_{ab}^2 = 0,$$
$$\left[Q_B^{(1)} \right]^2 = 0 \Leftrightarrow s_b Q_B^{(1)} = -i \left\{ Q_B^{(1)}, Q_B^{(1)} \right\} = 0 \Leftrightarrow s_b^2 = 0. \tag{79}$$

The above equation completes our discussion on the proof of the off-shell nilpotency of the conserved (anti-)BRST charges in the *ordinary* space.

Now, we dwell on the proof of the absolute anticommutativity (i.e., $Q_B^{(1)} Q_{\bar{B}}^{(1)} + Q_{\bar{B}}^{(1)} Q_B^{(1)} = 0$) of the conserved off-shell nilpotent (anti-)BRST charges [$Q_{(\bar{B})B}^{(1)}$]. Towards this central objective in mind, first of all, we assume the sanctity and validity of the CF-type restriction, $B + \bar{B} + i (\bar{C} \dot{C} - \dot{\bar{C}} C) = 0$, *right from the beginning*. As a result, we can express the (anti-)BRST charges $Q_{(\bar{B})B}^{(1)}$, in the alternative forms, as follows:

$$Q_{\bar{B}}^{(1)} \longrightarrow Q_{\bar{B}}^{(2)} = e^2 \left(B\dot{\bar{C}} - \dot{B}\bar{C} + 2i\dot{\bar{C}}\bar{C}\dot{C} \right) + 2ie\dot{e}\dot{\bar{C}}\bar{C}C,$$

$$Q_B^{(1)} \longrightarrow Q_B^{(2)} = e^2 \left(\dot{B}C - B\dot{C} - 2i\dot{C}\dot{C}C \right) - 2ie\dot{e}\bar{C}\dot{C}C. \tag{80}$$

We point out that it is because of the use of the CF-type restriction $[B + \bar{B} + i(\bar{C}\dot{C} - \dot{\bar{C}}C) = 0]$ that we have been able to express $Q_{\bar{B}}^{(2)}$ in terms of the Nakanishi-Lautrup auxiliary variable $B(\tau)$ and $Q_B^{(2)}$ in the language of other Nakanishi-Lautrup-type auxiliary variable $\bar{B}(\tau)$. At this crucial stage, we observe the following interesting relationships:

$$Q_B^{(2)} = s_{ab} \left[-ie^2\dot{C}C \right],$$
$$Q_{\bar{B}}^{(2)} = s_b \left[ie^2\dot{C}\bar{C} \right]. \tag{81}$$

In other words, we have been able to express the anti-BRST charge $[Q_{\bar{B}}^{(2)}]$ as the BRST *exact* quantity. On the other hand, we have been able to write the BRST charge $[Q_B^{(2)}]$ as an *exact* quantity w.r.t. the nilpotent anti-BRST transformation s_{ab}. A close and careful observation of (81) leads to the following (due to the well-known relationship between the continuous (anti-)BRST symmetry transformations $[s_{(a)b}]$ and their generators as conserved (anti-)BRST charges $[Q_{(\bar{B})B}^{(2)}]$), namely,

$$s_{ab} Q_B^{(2)} = -i\left\{ Q_B^{(2)}, Q_{\bar{B}}^{(2)} \right\} = 0 \Leftrightarrow s_{ab}^2 = 0,$$
$$s_b Q_{\bar{B}}^{(2)} = -i\left\{ Q_{\bar{B}}^{(2)}, Q_B^{(2)} \right\} = 0 \Leftrightarrow s_b^2 = 0. \tag{82}$$

As a result, we observe that the absolute anticommutativity of the (anti-)BRST charges $[Q_{(\bar{B})B}^{(2)}]$ is related to the nilpotency $[s_{(a)b}^2] = 0$ of the (anti-)BRST symmetries.

We would like to lay stress on the key results that have been seen in Equation (82). It is very interesting (due to the validity of the CF-type restriction on our theory) to pinpoint that (i) the anticommutativity of the BRST charge $Q_B^{(2)}$ *with* the anti-BRST charge $Q_{\bar{B}}^{(2)}$ is intimately connected with the nilpotency ($s_b^2 = 0$) of the BRST transformations (s_b) and (ii) the anticommutativity property of the anti-BRST charge $Q_{\bar{B}}^{(2)}$ *with* the BRST charge $Q_B^{(2)}$ owes its origin to the nilpotency ($s_{ab}^2 = 0$) of the anti-BRST transformations (s_{ab}). We conclude this subsection with the following remarks. First, we have shown that the nilpotency of the (anti-)BRST charges $[Q_{(\bar{B})B}]$ is deeply related with the nilpotency of the (anti-)BRST transformations $[s_{(a)b}]$. Second, we have been able to express the *modified* form of the BRST charge $[Q_B^{(1)}]$ and anti-BRST charge $Q_{\bar{B}}^{(1)}$ as the *exact* expressions w.r.t. the BRST transformations (s_b) and anti-BRST transformations (s_{ab}) (cf. Equation (78)), respectively. Third, it is due

to the *existence* of the CF-type restriction on our theory that we have been able to express *another* modified form of the BRST charge $Q_B^{(2)}$ as *an* exact expression w.r.t. the anti-BRST transformations (s_{ab}) *and* the anti-BRST charge $Q_{\bar{B}}^{(2)}$ in the BRST-*exact* form. This exercise has enabled us to prove the absolute anticommutativity (i.e., $\{Q_B^{(2)}, Q_{\bar{B}}^{(2)}\} = 0$) of the nilpotent (anti-)BRST charges $Q_{(\bar{B})B}^{(2)}$. Finally, the proof of the absolute anticommutativity property (cf. Equation (82)) *crucially* depends on the existence of the CF-type restriction. Thus, in a subtle manner, we have *derived* and *corroborated* the sanctity of the existence of the CF-type restriction $B + \bar{B} + i(\bar{C}\dot{C} - \dot{\bar{C}}C) = 0$ on our theory. We have provided an alternative proof for the appearance of the CF-type restriction (on our 1D reparameterization invariant SUSY theory) in our Appendix D. This completes our discussions on the absolute anticommutativity property of the conserved (anti-)BRST charges (in the *ordinary* Minkowski space).

7.2. Off-Shell Nilpotency and Absolute Anticommutativity Properties: ACSA to BRST Formalism in Superspace. In this subsection, we capture the properties of the nilpotency (i.e., fermionic nature) and absolute anticommutativity (i.e., linear independence) of the (anti-)BRST charges within the purview of ACSA where the superspace consideration on the $(1, 1)$-dimensional (anti-)chiral super submanifolds has been taken into account. First of all, we focus on the off-shell nilpotency $[Q_{(\bar{B})B}^2 = 0]$ of the (anti-)BRST charges $[Q_{(\bar{B})B}]$. In this context, keeping in our knowledge the mappings: $\partial_\theta \leftrightarrow s_{ab}$, $\partial_{\bar{\theta}} \leftrightarrow s_b$, it can be readily seen that the expressions for the (anti-)BRST charges that have been quoted in Equation (78) can be translated into the superspace as follows:

$$Q_{\bar{B}}^{(1)} = \frac{\partial}{\partial\theta} \left[iE^{(ab)}(\tau,\theta)E^{(ab)}(\tau,\theta)\left\{ \bar{F}^{(ab)}(\tau,\theta)\dot{F}^{(ab)}(\tau,\theta) - \dot{\bar{F}}^{(ab)}(\tau,\theta)F^{(ab)}(\tau,\theta) \right\} \right]$$
$$\equiv \int d\theta \left[iE^{(ab)}(\tau,\theta)E^{(ab)}(\tau,\theta)\left\{ \bar{F}^{(ab)}(\tau,\theta)\dot{F}^{(ab)}(\tau,\theta) - \dot{\bar{F}}^{(ab)}(\tau,\theta)F^{(ab)}(\tau,\theta) \right\} \right], \tag{83}$$

$$Q_B^{(1)} = \frac{\partial}{\partial\bar{\theta}} \left[iE^{(b)}(\tau,\bar{\theta})E^{(b)}(\tau,\bar{\theta})\left\{ \bar{F}^{(b)}(\tau,\bar{\theta})\dot{F}^{(b)}(\tau,\bar{\theta}) - \dot{\bar{F}}^{(b)}(\tau,\bar{\theta})F^{(b)}(\tau,\bar{\theta}) \right\} \right]$$
$$\equiv \int d\bar{\theta} \left[iE^{(b)}(\tau,\bar{\theta})E^{(b)}(\tau,\bar{\theta})\left\{ \bar{F}^{(b)}(\tau,\bar{\theta})\dot{F}^{(b)}(\tau,\bar{\theta}) - \dot{\bar{F}}^{(b)}(\tau,\bar{\theta})F^{(b)}(\tau,\bar{\theta}) \right\} \right], \tag{84}$$

where the supervariables with the superscripts (ab) and (b) have been obtained in Equations (64) and (57), respectively. At this stage, the off-shell nilpotency $[(Q_{(\bar{B})B}^{(1)})^2 = 0]$ of the conserved (anti-)BRST charges $[Q_{(\bar{B})B}^{(1)}]$ can be written in the superspace (by using the theoretical techniques and

tricks of ACSA to BRST formalism) as

$$\partial_\theta Q_{\bar{B}}^{(1)} = 0 \Leftrightarrow \partial_\theta^2 = 0,$$
$$\partial_{\bar{\theta}} Q_B^{(1)} = 0 \Leftrightarrow \partial_{\bar{\theta}}^2 = 0. \tag{85}$$

Thus, we conclude that the off-shell nilpotency of the anti-BRST charge $(Q_{\bar{B}}^{(1)})$ is deeply related to the nilpotency $(\partial_\theta^2 = 0)$ of the translational generator (∂_θ) along the θ-direction of the $(1, 1)$-dimensional *chiral* super submanifold of the *general* $(1, 2)$-dimensional supermanifold. Similar type of comments can be made in the context of the off-shell nilpotency of the BRST charge $Q_B^{(1)}$ and its intimate relationship with the nilpotency $(\partial_{\bar{\theta}}^2 = 0)$ of the translational generator $(\partial_{\bar{\theta}})$ on the *antichiral* super submanifold.

We concentrate now on capturing the absolute anticommutativity of the (anti-)BRST charges within the purview of ACSA where the superspace of the $(1, 1)$-dimensional (anti-)chiral super submanifolds is taken into consideration. Towards this central goal in our mind, we express the modified forms of the (anti-)BRST charges $Q_{(\bar{B})B}^{(2)}$ of Equation (81) in the following mathematical expression within the framework of ACSA, namely,

$$Q_{\bar{B}}^{(2)} = \frac{\partial}{\partial \bar{\theta}} \left[i E^{(b)} \left(\tau, \bar{\theta}\right) E^{(b)} \left(\tau, \bar{\theta}\right) \dot{\bar{F}}^{(b)} \left(\tau, \bar{\theta}\right) \bar{F}^{(b)} \left(\tau, \bar{\theta}\right) \right]$$
$$\equiv \int d\bar{\theta} \left[i E^{(b)} \left(\tau, \bar{\theta}\right) E^{(b)} \left(\tau, \bar{\theta}\right) \dot{\bar{F}}^{(b)} \left(\tau, \bar{\theta}\right) \bar{F}^{(b)} \left(\tau, \bar{\theta}\right) \right], \tag{86}$$

$$Q_B^{(2)} = \frac{\partial}{\partial \theta} \left[-i E^{(ab)}(\tau, \theta) E^{(ab)}(\tau, \theta) \dot{F}^{(ab)}(\tau, \theta) F^{(ab)}(\tau, \theta) \right]$$
$$\equiv \int d\theta \left[-i E^{(ab)}(\tau, \theta) E^{(ab)}(\tau, \theta) \dot{F}^{(ab)}(\tau, \theta) F^{(ab)}(\tau, \theta) \right], \tag{87}$$

where the supervariables with the superscripts (ab) and (b) have been quoted in Equations (64) and (57), respectively. At this crucial juncture, we note the following:

$$\partial_{\bar{\theta}} Q_{\bar{B}}^{(2)} = 0 \Leftrightarrow \partial_{\bar{\theta}}^2 = 0,$$
$$\partial_\theta Q_B^{(2)} = 0 \Leftrightarrow \partial_\theta^2 = 0. \tag{88}$$

The above relations are nothing but the explicit proof of the absolute anticommutativity relations of the conserved (anti-)BRST charges $Q_{(\bar{B})B}^{(2)}$ (within the ambit of ACSA).

We wrap up this subsection with the following comments. First, the off-shell nilpotency $[Q_{(\bar{B})B}^2 = 0]$ of the (anti-)BRST charges $Q_{(\bar{B})B}$ is intimately connected with the nilpotency $(\partial_\theta^2 = 0, \partial_{\bar{\theta}}^2 = 0)$ of the translational generators $(\partial_\theta, \partial_{\bar{\theta}})$ along the $(\theta, \bar{\theta})$-directions of the $(1, 1)$-dimensional *chiral* and *antichiral* super submanifolds. Second, in the *ordinary* space, the above statements of the off-shell nilpo-

tency are captured in Equations (75) and (79). Third, the absolute anticommutativity of the BRST charge *with* the anti-BRST charge is related to the nilpotency $(\partial_\theta^2 = 0)$ of the translational generator (∂_θ) along the θ-direction of the *chiral* super submanifold. The absolute anticommutativity of the anti-BRST charge *with* the BRST charge, on the other hand, is connected with the nilpotency $(\partial_{\bar{\theta}}^2 = 0)$ of the translational generator $(\partial_{\bar{\theta}})$ along the $\bar{\theta}$-direction of the *antichiral* super submanifold. Fourth, the above statements have been corroborated in the *ordinary* space by Equation (82) where the off-shell nilpotency $[s_{(a)b}^2 = 0]$ of the (anti-)BRST transformations $[s_{(a)b}]$ and the anticommutativity $[\{Q_B^{(2)}, Q_{\bar{B}}^{(2)}\} = 0]$ of the (anti-)BRST charges $[Q_{(\bar{B})B}^{(2)}]$ are found to be interconnected in an intimate and beautiful manner.

8. Conclusions

The USFA to BRST formalism (see, e.g., [10–12]) is useful in the context of the gauge theories where the spacetime coordinates do *not* change. Thus, it was a challenge to include the diffeomorphism (i.e., the general spacetime transformations) within the framework of Bonora-Tonin (BT) superfield approach to BRST formalism (see, e.g., [10–12]). This was achieved by Bonora in Ref. [22] which has been christened by us as the MBTSA where the generalization of the 1D diffeomorphism [i.e., $\tau \longrightarrow \tau' = f(\tau) \equiv \tau - \varepsilon(\tau)$] to the $(1, 2)$-dimensional supermanifold (cf. Equation (35)) has played an important role in the derivation of the (anti-)BRST symmetry transformations (cf. Equation (50)) for the target space variables $(x_\mu, p_\mu, \psi_\mu, \psi_5)$. In addition, this approach has enabled us to deduce the (anti-)BRST invariant CF-type restriction $B + \bar{B} + i(\bar{C}\dot{C} - \dot{\bar{C}}C) = 0$ that is responsible for the absolute anticommutativity of the (anti-)BRST symmetry transformations (cf. Equations (20) and (21)) and existence of the coupled (but equivalent) Lagrangians (24) for our theory.

We have taken into account the standard (anti-)BRST symmetry transformations $(s_b \bar{C} = iB, s_{ab} C = i\bar{B})$ for the (anti-)ghost variables $(\bar{C})C$ which have, in a subtle manner, forced us to consider the *(anti-)chiral* super expansions (cf. Equation (47)). This has provided us the clue to adopt the ACSA to BRST formalism for the deduction of the proper (anti-)BRST transformations for the *rest* of the variables of our theory (cf. Section 5). Within the purview of ACSA, we have derived the CF-type restriction when we have proven the *equivalence* of the coupled (but equivalent) Lagrangians (cf. Section 6). Furthermore, it is the validity of the CF-type restriction $B + \bar{B} + i(\bar{C}\dot{C} - \dot{\bar{C}}C) = 0$ that has enabled us to write (i) the BRST charge as an *exact* quantity w.r.t. the nilpotent anti-BRST transformation *and* (ii) the anti-BRST charge as an *exact* expression w.r.t. the nilpotent BRST transformation. These observations have been *responsible* for the proof of the absolute anticommutativity of the (anti-)BRST charges (cf. Section 7). In other words, it is the *proof* of the anticommutativity of the conserved and

nilpotent charges $[Q^{(2)}_{(\bar{B})B}]$ which lead to the existence of the CF-type restriction on our SUSY theory (cf. Section 7).

We would like to emphasize that the *observation* of the absolute anticommutativity property, in the context of the conserved (anti-)BRST charges, is a novel observation because of the fact that *only* the (anti-)chiral super expansions have been considered within the ambit of ACSA. This observation of the absolute anticommutativity property is obvious when one takes into account the *full* super expansions of the supervariables *along all* the possible Grassmannian directions of the suitably chosen supermanifold on which the *ordinary* theory is generalized. Furthermore, the appearance of the CF-type restriction in the computations of $s_b L_{\bar{B}}$ and $s_{ab} L_B$ (cf. Equations (28) and (29)) in the *ordinary* space *and* its analogue in the *superspace* are very interesting observations in our present endeavor (cf. Section 6). The other observation that merits a clear and special mention is the *universality* of the CF-type restriction $B + \bar{B} + i(\bar{C}\dot{C} - \dot{\bar{C}}C) = 0$ in the context of the reparameterization (i.e., 1D diffeomorphism) invariant non-SUSY theory of a scalar relativistic particle as well as a nonrelativistic particle [23, 24] and our present SUSY system of a spinning relativistic particle (where the nilpotent SUSY transformations exist between the specific set of bosonic and fermionic variables of our SUSY theory).

It is worthwhile to mention that, for the D-dimensional diffeomorphism invariant theory [22, 25] where the infinitesimal diffeomorphism symmetry transformation is $x_\mu \longrightarrow x'_\mu = x_\mu - \varepsilon_\mu(x)$ (with $\mu = 0, 1, 2 \cdots D - 1$), the *general* form of the CF-type restriction has been obtained as $B_\mu + \bar{B}_\mu + i(\bar{C}^\rho \partial_\rho C_\mu + C^\rho \partial_\rho \bar{C}_\mu) = 0$ where the (anti-)ghost fields $(\bar{C}_\mu)C_\mu$ correspond to the infinitesimal transformation parameter $\varepsilon_\mu(x)$ in the general coordinate transformation $x'_\mu = x_\mu - \varepsilon_\mu(x)$ *and* the Nakanishi-Lautrup fields $(\bar{B}_\mu)B_\mu$ appear in the (anti-)BRST symmetry transformations: $s_b \bar{C}_\mu = i B_\mu$, $s_{ab} C_\mu = i \bar{B}_\mu$. It is straightforward to note that the CF-type restriction $B + \bar{B} + i(\bar{C}\dot{C} - \dot{\bar{C}}C) = 0$ is the limiting case of the above *general* D-dimensional CF-type restriction in the case of the BRST approach to D-dimensional diffeomorphism invariant theory [22, 25]. Thus, our theoretical treatments of the reparameterization (i.e., 1D diffeomorphism) invariant theories of the *scalar* and *spinning* relativistic particles are correct.

One of the highlights of ACSA to BRST formalism is the observation that it distinguishes between the suitably chosen (1, 1)-dimensional *chiral* and *antichiral* super submanifolds in the proof of the absolute anticommutativity of the conserved (anti-)BRST charges. For instance, we note that the anticommutativity of the BRST charge $[Q^{(2)}_B]$ *with* the anti-BRST charge $[Q^{(2)}_{\bar{B}}]$ is connected with the nilpotency ($\partial^2_\theta = 0$) of the translational generator (∂_θ) along the θ-direction of the *chiral* super submanifold (cf. Equation (88)). On the other hand, the anticommutativity of the anti-BRST charge $[Q^{(2)}_{\bar{B}}]$ *with* the BRST charge $[Q^{(2)}_B]$ crucially depends on the

nilpotency ($\partial^2_{\bar{\theta}} = 0$) of the translational generator ($\partial_{\bar{\theta}}$) (cf. Equation (88)) along the $\bar{\theta}$-direction of the *antichiral* super submanifold (cf. Section 7 for details). This observation is a reflection of our discussion on the absolute anticommutativity property of the (anti-)BRST charges in the *ordinary* space (cf. Section 7) where the off-shell nilpotency of the (anti-)BRST transformations (cf. Equation (82)) plays a decisive role.

We plan to extend our present study to the physical (3 + 1)-dimensional (4D) theories of the gravitation and higher dimensional (super)string theories where there is existence of the diffeomorphism invariance. In other words, we plan to apply the ideas of MBTSA and ACSA *together* to find out the (anti-)BRST symmetries of the above mentioned theories. The mathematical elegance, rigor, and beauty of the MBTSA [22, 25] should find more applications to some physical systems of interest in theoretical high-energy physics. We envisage to take up these challenges in our future investigations. Before we end this section, it is worthwhile to point out that in our earlier works (see, e.g., [27, 28]), we have applied the techniques and tricks of ACSA to obtain the nilpotent symmetries of the $N = 2$ SUSY quantum mechanical models of interest. However, we have found that the conserved charges are *not* absolutely anticommuting. Thus, our observation of the absolute anticommutativity property in the context of (anti-)BRST charges is *novel* and interesting.

Appendix

A. On the Step-by-Step Computation of the Secondary Variables for the Off-Shell Nilpotent Anti-BRST Transformations

In this appendix, we concentrate on the clear-cut derivation of the secondary variables (cf. Equation (63)) in terms of the basic and auxiliary variables of the Lagrangians (24). For this purpose, we invoke the basic principle of ACSA which states that the anti-BRST invariant expressions (cf. Equation (59)) should not depend on θ (i.e., Grassmannian variable) when these quantities are promoted onto the (1, 1)-dimensional *chiral* version of super submanifold. First of all, we consider $s_{ab}(\bar{C}\dot{x}_\mu) = 0$ which leads to the following restriction:

$$\bar{F}(\tau, \theta)\dot{X}^{(hc)}_\mu(\tau, \theta) = \bar{C}(\tau)\dot{x}_\mu(\tau). \tag{A.1}$$

At this stage, we substitute the super expansions from (62) and (58) which leads to the precise determination of the secondary variable $\bar{b}_3 = \bar{C}\dot{\bar{C}}$. As a consequence, we have now the super expansion of the chiral supervariable $\bar{F}(\tau, \theta)$ as

$$\bar{F}^{(ab)}(\tau, \theta) = \bar{C}(\tau) + \theta\left(\bar{C}\dot{\bar{C}}\right) \equiv \bar{C}(\tau) + \theta\left(s_{ab}\bar{C}\right), \tag{A.2}$$

where the superscript (ab) denotes the *chiral* super expansion of $\bar{F}(\tau, \theta)$ after the application of the anti-BRST restriction (A.1). We find that the coefficient of θ is the anti-BRST

symmetry transformation of $\bar{C}$ (cf. Equation (20)). In other words, we find that $\partial_\theta \bar{F}^{(ab)}(\tau,\theta) = s_{ab}\bar{C}$ which agrees with the mapping $\partial_\theta \leftrightarrow s_{ab}$ of Refs. [10–12].

At this juncture, we take up the anti-BRST invariant quantities (i.e., $s_{ab}[d/d\tau(\bar{C}e)] = 0$, $s_{ab}[d/d\tau(\bar{C}\chi)] = 0$) which leads to the following restrictions:

$$\frac{d}{d\tau}\left[\bar{F}^{(ab)}(\tau,\theta)E(\tau,\theta)\right] = \frac{d}{d\tau}\left[\bar{C}(\tau)e(\tau)\right],$$
$$\frac{d}{d\tau}\left[\bar{F}^{(ab)}(\tau,\theta)K(\tau,\theta)\right] = \frac{d}{d\tau}\left[\bar{C}(\tau)\chi(\tau)\right]. \tag{A.3}$$

Substitutions from (A.2) and (62) lead to the following equations in terms of the fermionic (anti-)ghost variables as well as secondary variables $\bar{f}_2$ and $\bar{b}_1$, namely,

$$\dot{\bar{C}}\bar{f}_2 + \bar{C}\dot{\bar{f}}_2 - \bar{C}\ddot{\bar{C}}e - \bar{C}\dot{\bar{C}}\dot{e} = 0,$$
$$\dot{\bar{C}}\bar{b}_1 + \bar{C}\dot{\bar{b}}_1 - \bar{C}\ddot{\bar{C}}\chi - \bar{C}\dot{\bar{C}}\dot{\chi} = 0. \tag{A.4}$$

It is straightforward to verify that we have the following solutions:

$$\bar{f}_2 = \bar{C}\dot{e} + \dot{\bar{C}}e,$$
$$\bar{b}_1 = \bar{C}\dot{\chi} + \dot{\bar{C}}\chi. \tag{A.5}$$

We take now the anti-BRST invariance: $s_{ab}(\dot{B}\bar{C} - B\dot{\bar{C}}) = 0$. This leads to the following restrictions on the chiral supervariables:

$$\dot{\tilde{B}}(\tau,\theta)\bar{F}^{(ab)}(\tau,\theta) - \tilde{B}(\tau,\theta)\dot{\bar{F}}^{(ab)}(\tau,\theta) = \dot{B}(\tau)\bar{C}(\tau) - B(\tau)\dot{\bar{C}}(\tau). \tag{A.6}$$

Substitutions of the super expansions from (62) and (A.2) lead to the following relationship:

$$\dot{B}\bar{C}\dot{\bar{C}} - B\bar{C}\ddot{\bar{C}} + \dot{\bar{f}}_1\bar{C} - \bar{f}_1\dot{\bar{C}} = 0. \tag{A.7}$$

It is evident that the precise solution is $\bar{f}_1 = \dot{B}\bar{C} - B\dot{\bar{C}}$. Thus far, we have been able to determine the precise forms of the secondary variables $(\bar{b}_1, \bar{b}_3, \bar{f}_1, \bar{f}_3)$ in terms of the basic, auxiliary, and (anti-)ghost variables of the Lagrangians (24).

In our present appendix, we have followed the tricks and techniques of ACSA to BRST formalism which was motivated by our standard assumption that $s_b\bar{C} = iB$, $s_{ab}C = i\bar{B}$ in the BRST approach to gauge and/or diffeomorphism invariant theories. This standard assumption led to the (anti-)chiral super expansions of the supervariables in Equation (47). This implies that we have already determined the remaining secondary variable of Equation (62) as $\bar{b}_2 = i\bar{B}$. This completes our discussion on the step-by-step determination of the secondary variables of (cf. Equation (63)) which are present in the chiral super expansions (62).

B. On the Proof of Equations (28) and (74) in the Ordinary Space

For the sake of completeness, we provide here the explicit proof of Equations (28) and/or (74) in the ordinary space that leads to the derivation of the CF-type restriction $B + \bar{B} + i(\bar{C}\dot{C} - \dot{\bar{C}}C) = 0$ which demonstrates, thereby, the equivalence of the Lagrangians L_B and $L_{\bar{B}}$ w.r.t. the (anti-)BRST symmetry transformations. This is due to our earlier observation (cf. Equation (27)) that $L_{\bar{B}}$ has a perfect symmetry w.r.t. the anti-BRST symmetry transformations s_{ab}. The direct applications of s_b on $L_{\bar{B}}$ lead to the following:

$$s_b L_{\bar{B}} = \frac{d}{d\tau}\left[CL_f - e^2\left(\bar{B}\dot{C} + i\dot{\bar{C}}\dot{C}C\right) - e\dot{e}\left(\bar{B}C + i\bar{C}\dot{C}C\right)\right]$$
$$+ e^2\left[\dot{\bar{B}}\dot{C} + \dot{B}\dot{C} + i\bar{C}\ddot{C}\dot{C} - i\ddot{\bar{C}}\dot{C}\dot{C}\right] + e\dot{e}\left[B\dot{C} + \bar{B}\dot{C} - i\dot{\bar{C}}\dot{C}\dot{C}\right]$$
$$+ \bar{B}\dot{\bar{B}}C - \bar{B}\dot{B}C - \bar{B}^2\dot{C} - 2B\bar{B}\dot{C} - i\dot{B}\bar{C}\dot{C}C + iB\dot{\bar{C}}\dot{C}C$$
$$+ 2i\dot{\bar{B}}\bar{C}\dot{C}C + 2i\bar{B}\bar{C}\ddot{C}C + 2i\bar{B}\dot{\bar{C}}\dot{C}C. \tag{B.1}$$

It is straightforward to note (with the input $\dot{C}^2 = 0$) that the coefficients of e^2 and $e\dot{e}$ can be expressed in terms of the CF-type restrictions $[B + \bar{B} + i(\bar{C}\dot{C} - \dot{\bar{C}}C) = 0]$ as follows:

$$e^2\left[\frac{d}{d\tau}\left\{B + \bar{B} + i\left(\bar{C}\dot{C} - \dot{\bar{C}}C\right)\right\}\right]\dot{C} + e\dot{e}\left[B + \bar{B} + i\left(\bar{C}\dot{C} - \dot{\bar{C}}C\right)\right]\dot{C}. \tag{B.2}$$

At this stage, we focus on the terms $\bar{B}\dot{\bar{B}}C - \bar{B}\dot{B}C - \bar{B}^2\dot{C} - 2B\bar{B}\dot{C}$ (from (B.1)) which can be expressed as a sum of a total derivative and other terms, namely,

$$\frac{d}{d\tau}\left[\bar{B}^2C\right] - \left(\dot{B} + \dot{\bar{B}}\right)\bar{B}C - 2\bar{B}^2\dot{C} - 2B\bar{B}\dot{C}$$
$$= \frac{d}{d\tau}\left[\bar{B}^2C\right] - \left[\dot{B} + \dot{\bar{B}} + i\left(\bar{C}\ddot{C} - \ddot{\bar{C}}C\right)\right]\bar{B}C + i\bar{B}\bar{C}\ddot{C}C$$
$$- 2\bar{B}\dot{C}\left[B + \bar{B} + i\left(\bar{C}\dot{C} - \dot{\bar{C}}C\right)\right] + 2i\bar{B}\dot{\bar{C}}\dot{C}C. \tag{B.3}$$

Now, adding the left-over terms without the total derivative terms as well as the CF-type terms from (B.1) and (B.3), we obtain the following:

$$2i\bar{B}\bar{C}\ddot{C}C - i\dot{B}\bar{C}\dot{C}C + i\dot{B}\dot{\bar{C}}\dot{C}C + 2i\dot{\bar{B}}\bar{C}\dot{C}C + i\bar{B}\bar{C}\ddot{C}C$$
$$+ 2i\bar{B}\dot{\bar{C}}\dot{C}C + 2i\dot{\bar{B}}\bar{C}\dot{C}C. \tag{B.4}$$

It can be readily seen that the following is completely true if we express the first two terms $(2i\bar{B}\bar{C}\ddot{C}C - i\dot{B}\bar{C}\dot{C}C$

) of the above equation, namely,

$$\frac{d}{d\tau}\left[2\,i\,\bar{B}\bar{C}\,\dot{C}\,C - i\,B\,\bar{C}\dot{C}\,C\right] - 2\,i\,\dot{\bar{B}}\bar{C}\,\dot{C}\,C - 2\,i\,\dot{\bar{B}}\,\bar{C}\,\dot{C}\,C + i\,B\,\dot{\bar{C}}\,\dot{C}\,C$$
$$+\,i\,B\,\bar{C}\,\ddot{C}\,C. \tag{B.5}$$

At this juncture, we add the terms from (B.4) and non-derivative terms from (B.5) to yield the following result:

$$2\,i(B+\bar{B})\,\dot{\bar{C}}\,\dot{C}\,C + i\,(B+\bar{B})\,\bar{C}\ddot{C}\,C, \tag{B.6}$$

which can be reexpressed, in terms of the CF-type restriction, as follows:

$$2\,i\left[B+\bar{B}+i\left(\bar{C}\dot{C}-\dot{\bar{C}}C\right)\right]\dot{\bar{C}}\dot{C}\,C + i\left[B+\bar{B}+i\left(\bar{C}\dot{C}-\dot{\bar{C}}C\right)\right]\bar{C}\ddot{C}\,C. \tag{B.7}$$

The total sum of the contributions from (B.3), (B.5), and (B.7) is equal to

$$\frac{d}{d\tau}\left[i(2\,\bar{B}-B)\bar{C}\,\dot{C}\,C + \bar{B}^2\,C\right] - \frac{d}{d\tau}\left[B+\bar{B}+i\left(\bar{C}\dot{C}-\dot{\bar{C}}C\right)\right]\bar{B}\,C$$
$$+\left[B+\bar{B}+i\left(\bar{C}\dot{C}-\dot{\bar{C}}C\right)\right]\left[2\,i\,\dot{\bar{C}}\dot{C}\,C + i\,\bar{C}\ddot{C}\,C - 2\,\bar{B}\dot{C}\right]. \tag{B.8}$$

Now, adding all the terms of Equations (B.1), (B.2), and (B.8), we obtain the *same* result as given in Equations (28) and/or (74) in the *ordinary* space for the computation of $s_b L_{\bar{B}}$.

We wrap up this appendix with the following concluding remarks. We have already demonstrated the existence of the CF-type restriction in the *ordinary space* (cf. Equation (28)) and *superspace* (cf. Equation (74)) by expressing the BRST symmetry transformation (i.e., $s_b L_{\bar{B}}$) of $L_{\bar{B}}$. Exactly in a similar manner, it can be shown that the quantity $s_{ab} L_B$ can be expressed in the *ordinary* space (cf. Equation (29)) and superspace (cf. Equation (72)) leading to the appearance of the CF-type restriction. It is interesting, furthermore, to point out that *this* restriction also appears when we prove the absolute anticommutativity of the conserved and off-shell nilpotent (anti-)BRST charges in the *ordinary* space as well as in the *superspace* (cf. Appendix D for details) using the ACSA to BRST formalism.

C. On the Different Forms of the Conserved (Anti-)BRST Charges

Besides the expressions for the (anti-)BRST charges in (31) and (32), we require different forms of these charges to prove their nilpotency (i.e., fermionic nature) and anticommutativity (i.e., linear independence) in a straightforward manner. This exercise has been found to be advantageous in the context of our discussion on ACSA, too. First and foremost, we concentrate on the derivation of the BRST charge in (77) and its usefulness. In this connection, we note

that the Noether charge (31) can be rewritten as follows:

$$Q_B \longrightarrow \tilde{Q}_B = e^2\left[B\dot{C} - \dot{B}\,C - i\,\dot{\bar{C}}\,\dot{C}\,C\right] + B^2\,C + e\,\dot{e}\,B\,C - i\,B\,\bar{C}\,\dot{C}\,C, \tag{C.1}$$

where we have used the following EL-EOMs (and their modified version), namely,

$$p_\mu\,\psi^\mu = m\,\psi_5,$$
$$\frac{e\,C}{2}\left(p^2 - m^2\right) = -e^2\,\dot{B}\,C - i\,e^2\,\dot{\bar{C}}\,\dot{C}\,C. \tag{C.2}$$

At this juncture, we note that $B = -e\,\dot{e} + i\,(2\,\dot{\bar{C}}\,C + \bar{C}\,\dot{C})$. The substitution of this expression leads to the following observations, namely,

$$B^2\,C + e\,\dot{e}\,B\,C = -i\,e\,\dot{e}\,\bar{C}\,\dot{C}\,C,$$
$$-i\,B\,\bar{C}\,\dot{C}\,C = i\,e\,\dot{e}\,\bar{C}\,\dot{C}\,C. \tag{C.3}$$

Thus, we obtain the modified form of the BRST charge $(\tilde{Q}_B)$ as

$$\tilde{Q}_B \longrightarrow \tilde{Q}_B^{(1)} = e^2\left[B\dot{C} - \dot{B}\,C - i\,\dot{\bar{C}}\,\dot{C}\,C\right]. \tag{C.4}$$

This is because of the fact that the sum of last *three* terms in (C.1) is zero. It can be readily checked that the above form of the charge (cf. Equation (C.4)) is *not* off-shell nilpotent. Hence, it is *not* suitable for our further discussions.

Let us focus on the derivation of an alternative form of this BRST charge. In the expression for the charge $\tilde{Q}_B$ (cf. Equation (C.1)), we can express $[-i\,B\,\bar{C}\,\dot{C}\,C]$, from the *third* equation from the top in (76) which yields the following:

$$-i\,B\,\bar{C}\,\dot{C}\,C = i\,e^2\,\bar{C}\,\ddot{C}\,C + 3\,i\,e\,\dot{e}\,\bar{C}\,\dot{C}\,C. \tag{C.5}$$

Using (C.3) and (C.5), we obtain

$$\tilde{Q}_B^{(1)} \longrightarrow \tilde{Q}_B^{(2)} = e^2\left[B\dot{C} - \dot{B}\,C + i\left(\bar{C}\,\ddot{C}\,C - \dot{\bar{C}}\,\dot{C}\,C\right)\right] + 2\,i\,e\,\dot{e}\,\bar{C}\,\dot{C}\,C. \tag{C.6}$$

It is elementary exercise to check that the above expression for the BRST charge is *still* not off-shell nilpotent of order two. Hence, it *cannot* be expressed as an exact quantity w.r.t. the BRST transformations (s_b). At this crucial point, we reexpress (C.5) in a different type of mathematical form as follows:

$$i\,B\,\bar{C}\,\dot{C}\,C = -i\,e^2\,\bar{C}\,\ddot{C}\,C - 2\,i\,e\,\dot{e}\,\bar{C}\,\dot{C}\,C - i\,e\,\dot{e}\,\bar{C}\,\dot{C}\,C. \tag{C.7}$$

From the relationship: $B = -e\,\dot{e} + i\,(2\,\dot{\bar{C}}\,C + \bar{C}\,\dot{C})$, it is clear that $i\,\bar{C}\,\dot{C} = B + e\,\dot{e} - 2\,i\,\dot{\bar{C}}\,C$. As a result, we have the

following:

$$i B \bar{C} \dot{C} C = B \left(B + e \dot{e} - 2 i \dot{\bar{C}} C \right) C \equiv B^2 C + e \dot{e} B C. \quad \text{(C.8)}$$

Substituting the above equality into (C.7), we obtain

$$B^2 C + e \dot{e} B C + i e \dot{e} \bar{C} \dot{C} C = -i e^2 \bar{C} \ddot{C} C - 2 i e \dot{e} \bar{C} \dot{C} C. \quad \text{(C.9)}$$

However, we also note that $i e \dot{e} \bar{C} \dot{C} C = -i B \bar{C} \dot{C} C$ (cf. Equation (C.3)) due to the fact that $B = -e \dot{e} + i \left(2 \dot{\bar{C}} C + \bar{C} \dot{C} \right)$. Thus, we observe that the equality in (C.9) reduces to

$$B^2 C + e \dot{e} B C - i B \bar{C} \dot{C} C = -i e^2 \bar{C} \ddot{C} C - 2 i e \dot{e} \bar{C} \dot{C} C. \quad \text{(C.10)}$$

The above equation leads to the following:

$$\tilde{Q}_B \longrightarrow Q_B^{(1)} = e^2 \left[B \dot{C} - \dot{B} C - i \left(\dot{\bar{C}} \dot{C} C + \bar{C} \ddot{C} C \right) \right] - 2 i e \dot{e} \bar{C} \dot{C} C, \quad \text{(C.11)}$$

where $\tilde{Q}_B$ is quoted in (C.1) and $Q_B^{(1)}$ is written in Equation (77) of the main body of our text. The importance of (C.11) is the observation that it can be written as an exact quantity w.r.t. the BRST transformations (s_b). As a result, the charge $Q_B^{(1)}$ is off-shell nilpotent of order two (i.e., $s_b Q_B^{(1)} = -i \{ Q_B^{(1)}, Q_B^{(1)} \} = 0 \Rightarrow [Q_B^{(1)}]^2 = 0$). Exactly similar kinds of arguments can be provided for the derivation of $Q_{\bar{B}}^{(1)}$ in Equation (77). As a consequence, we observe that $s_{ab} Q_{\bar{B}}^{(1)} = -i \{ Q_{\bar{B}}^{(1)}, Q_{\bar{B}}^{(1)} \} = 0 \Rightarrow [Q_{\bar{B}}^{(1)}]^2 = 0$. This observation proves the off-shell nilpotency of $Q_{\bar{B}}^{(1)}$ in a straightforward fashion.

D. On an Alternative Proof of the Existence of the CF-Type Restriction on Our SUSY System

Our present appendix provides an alternative proof of the existence of the CF-type restriction $B + \bar{B} + i \left(\bar{C} \dot{C} - \dot{\bar{C}} C \right) = 0$ which is straightforward *and* different from our derivation in Section 7 where the proof is a bit *subtle*. In this context, we apply, first of all, directly the BRST symmetry transformations (s_b) on the expression for the anti-BRST charge $Q_{\bar{B}}^{(1)}$ (cf. Equation (77)). In other words, we derive $s_b Q_{\bar{B}}^{(1)} = -i \{ Q_{\bar{B}}^{(1)}, Q_B^{(1)} \}$ which is nothing but the anticommutator of the anti-BRST charge ($Q_{\bar{B}}^{(1)}$) *with* the BRST charge ($Q_B^{(1)}$

). The outcome of this exercise can be explicitly expressed as

$$\begin{aligned}
s_b Q_{\bar{B}}^{(1)} = {} & 2 e^2 \bar{B} \dot{C} C - 2 e^2 \dot{\bar{B}} \bar{C} \dot{C} - 2 e \dot{e} \dot{\bar{B}} \bar{C} C + 2 e \dot{e} \bar{B} \dot{\bar{C}} C \\
& - e^2 \ddot{\bar{B}} \bar{C} C + e^2 \bar{B} \bar{C} \ddot{C} + 2 i e \dot{e} \dot{\bar{C}} \bar{C} \dot{C} C + e^2 B \ddot{\bar{C}} C \\
& - e^2 \ddot{B} \bar{C} C + 2 e \dot{e} B \dot{\bar{C}} C - 2 e \dot{e} \dot{B} \bar{C} C + i e^2 \dot{\bar{C}} \bar{C} \dot{C} C \\
& + i e^2 \bar{C} \dot{\bar{C}} \dot{C} C + e^2 B \dot{\bar{C}} \dot{C} - e^2 \dot{B} \bar{C} \dot{C} + i e^2 B \dot{\bar{B}} \\
& - i e^2 \bar{B} \dot{B} + e^2 \dot{\bar{B}} \bar{C} C - e^2 \bar{B} \dot{\bar{C}} \dot{C}.
\end{aligned}$$
$$\text{(D.1)}$$

The above expression can be rearranged in such a manner that we shall have the coefficients of e^2 and $2 e \dot{e}$ separately and independently as illustrated below:

$$\begin{aligned}
s_b Q_{\bar{B}}^{(1)} = {} & 2 e \dot{e} \left[B \dot{\bar{C}} C + \bar{B} \dot{\bar{C}} C - \dot{B} \bar{C} C - \dot{\bar{B}} \bar{C} C - i \dot{\bar{C}} \bar{C} \dot{C} C \right] \\
& + e^2 \left[2 \left(\bar{B} \dot{\bar{C}} - \dot{\bar{B}} \bar{C} \right) \dot{C} - i \left(\ddot{\bar{C}} \dot{C} + \dot{\bar{C}} \ddot{C} \right) \bar{C} C + \left(\ddot{B} C - \bar{B} \ddot{C} \right) \bar{C} \right. \\
& \left. - \left(\ddot{\bar{B}} C - \bar{B} \dot{C} \right) \dot{\bar{C}} + i B \dot{\bar{B}} - i \bar{B} \dot{B} + \left(\bar{B} \ddot{\bar{C}} - \ddot{\bar{B}} \bar{C} \right) C + \left(B \dot{\bar{C}} - \dot{B} \bar{C} \right) \dot{C} \right].
\end{aligned}$$
$$\text{(D.2)}$$

At this stage, the straightforward algebraic exercise produces the following result:

$$\begin{aligned}
s_b Q_{\bar{B}}^{(1)} = {} & -i \left\{ Q_{\bar{B}}^{(1)}, Q_B^{(1)} \right\} \\
\equiv {} & 2 e \dot{e} \left[\left\{ B + \bar{B} + i \left(\bar{C} \dot{C} - \dot{\bar{C}} C \right) \right\} \dot{\bar{C}} C - \frac{d}{d\tau} \left\{ B + \bar{B} + i \left(\bar{C} \dot{C} - \dot{\bar{C}} C \right) \right\} \bar{C} C \right] \\
& + e^2 \left[\frac{d}{d\tau} \left\{ \left[B + \bar{B} + i \left(\bar{C} \dot{C} - \dot{\bar{C}} C \right) \right] \left(\dot{\bar{C}} C - i \bar{B} \right) \right. \right. \\
& \left. \left. - \frac{d}{d\tau} \left[B + \bar{B} + i \left(\bar{C} \dot{C} - \dot{\bar{C}} C \right) \right] \bar{C} C \right\} + 2 i \dot{\bar{B}} \left[B + \bar{B} + i \left(\bar{C} \dot{C} - \dot{\bar{C}} C \right) \right] \right],
\end{aligned}$$
$$\text{(D.3)}$$

which demonstrates *explicitly* that the absolute anticommutativity property ($\{ Q_{\bar{B}}^{(1)}, Q_B^{(1)} \} = 0$) is satisfied if and only if the CF-type restriction $B + \bar{B} + i \left(\bar{C} \dot{C} - \dot{\bar{C}} C \right) = 0$ is invoked from *outside*. In other words, the *requirement* of the absolute anticommutativity of the (anti-)BRST charges leads to the existence of a CF-type restriction on our theory.

We now concentrate on the computation of $s_{ab} Q_B^{(1)} = -i \{ Q_B^{(1)}, Q_{\bar{B}}^{(1)} \}$ which leads to the following as the sum of the coefficients of e^2 and $2 e \dot{e}$, namely,

$$\begin{aligned}
s_{ab} Q_B^{(1)} = {} & 2 e \dot{e} \left[B \bar{C} \dot{C} - \dot{B} \bar{C} C + \bar{B} \bar{C} \dot{C} - \dot{\bar{B}} \bar{C} C - i \dot{C} \bar{C} \dot{C} C \right] \\
& + e^2 \left[2 B \dot{\bar{C}} \dot{C} - 2 \dot{B} \bar{C} \dot{C} - i \dot{C} \bar{C} \ddot{C} C - i \ddot{C} \bar{C} \dot{C} C - \ddot{B} \bar{C} C \right. \\
& + B \ddot{\bar{C}} C + \dot{B} \bar{C} \dot{C} - B \dot{\bar{C}} \dot{C} - i \dot{B} \bar{B} + i B \dot{\bar{B}} + \bar{B} \dot{\bar{C}} \dot{C} - \dot{\bar{B}} \bar{C} C \\
& \left. + \bar{B} \bar{C} \ddot{C} - \ddot{\bar{B}} \bar{C} C \right].
\end{aligned}$$
$$\text{(D.4)}$$

The above expression can be rearranged by performing some algebraic tricks in the following form (in terms of the

CF-type restrictions $B + \bar{B} + i(\bar{C}\dot{C} - \dot{\bar{C}}C) = 0$), namely,

$$
\begin{aligned}
s_{ab}\, Q_B^{(1)} = e^2 \bigg[&\frac{d}{d\tau}\Big\{ \big[B + \bar{B} + i\big(\bar{C}\dot{C} - \dot{\bar{C}}C\big)\big]\big(i B + \bar{C}\dot{C}\big) \\
&- \frac{d}{d\tau}\big[B + \bar{B} + i\big(\bar{C}\dot{C} - \dot{\bar{C}}C\big)\big]\bar{C}C\Big\} \\
&- 2\, i\, \dot{B}\,\big[B + \bar{B} + i\big(\bar{C}\dot{C} - \dot{\bar{C}}C\big)\big]\bigg] \\
&+ 2\, e\, \dot{e}\, \bar{C}\Big[\big(B + \bar{B} + i\big(\bar{C}\dot{C} - \dot{\bar{C}}C\big)\big)\dot{C} \\
&- \frac{d}{d\tau}\big[B + \bar{B} + i\big(\bar{C}\dot{C} - \dot{\bar{C}}C\big)\big]C\Big].
\end{aligned}
\tag{D.5}
$$

The above expression demonstrates that the absolute anticommutativity property (i.e., $s_{ab}\, Q_B^{(1)} = -i\{Q_B^{(1)}, Q_{\bar{B}}^{(1)}\} = 0$) is satisfied if and only if the validity of the CF-type restriction is invoked at the *quantum* level in our BRST *quantized* theory. There is another way of stating this observation. That is to say the fact that the sanctity of anticommutativity property (i.e., $\{Q_B^{(1)}, Q_{\bar{B}}^{(1)}\} = 0$) leads to the deduction of the CF-type restriction in a straightforward manner.

We close our appendix with the final remark that we can capture the derivation of the CF-type restriction within the purview of ACSA. Towards this central objective in our mind, first of all, we generalize the expressions for the BRST and anti-BRST charges (cf. Equation (77)) on the $(1, 1)$-dimensional *chiral* and *antichiral* supermanifold as

$$
\begin{aligned}
Q_B^{(1)} \longrightarrow \tilde{Q}_B^{(1c)}(\tau,\theta) = E^{(ab)}(\tau,\theta) E^{(ab)}(\tau,\theta)\Big[\tilde{B}^{(ab)}(\tau,\theta)\dot{F}^{(ab)}(\tau,\theta) \\
- \dot{\tilde{B}}^{(ab)}(\tau,\theta) F^{(ab)}(\tau,\theta) - i\Big(\bar{F}^{(ab)}(\tau,\theta)\ddot{F}^{(ab)}(\tau,\theta) F^{(ab)}(\tau,\theta) \\
+ \dot{\bar{F}}^{(ab)}(\tau,\theta)\dot{F}^{(ab)}(\tau,\theta) F^{(ab)}(\tau,\theta) \Big) \Big] \\
- 2\, i\, E^{(ab)}(\tau,\theta)\dot{E}^{(ab)}(\tau,\theta)\,\bar{F}^{(ab)}(\tau,\theta)\,\dot{F}^{(ab)}(\tau,\theta)\, F^{(ab)}(\tau,\theta),
\end{aligned}
$$

$$
\begin{aligned}
Q_{\bar{B}}^{(1)} \longrightarrow \tilde{Q}_{\bar{B}}^{(1ac)}(\tau,\bar{\theta}) = E^{(b)}(\tau,\bar{\theta}) E^{(b)}(\tau,\bar{\theta})\Big[\dot{\tilde{\bar{B}}}^{(b)}(\tau,\bar{\theta}) \bar{F}^{(b)}(\tau,\bar{\theta}) \\
- \tilde{\bar{B}}^{(b)}(\tau,\bar{\theta})\dot{F}^{(b)}(\tau,\bar{\theta}) + i\Big(\ddot{F}^{(b)}(\tau,\bar{\theta})\bar{F}^{(b)}(\tau,\bar{\theta}) F^{(b)}(\tau,\bar{\theta}) \\
+ \dot{F}^{(b)}(\tau,\bar{\theta})\bar{F}^{(b)}(\tau,\bar{\theta})\dot{F}^{(b)}(\tau,\bar{\theta}) \Big) \Big] \\
+ 2\, i\, E^{(b)}(\tau,\bar{\theta})\dot{E}^{(b)}(\tau,\bar{\theta})\dot{F}^{(b)}(\tau,\bar{\theta})\bar{F}^{(b)}(\tau,\bar{\theta}) F^{(b)}(\tau,\bar{\theta}),
\end{aligned}
\tag{D.6}
$$

where the superscripts $(1c)$ and $(1ac)$, on the BRST and anti-BRST charges, denote the *chiral* and *antichiral* versions of Equation (77). The other notations have already been explained earlier. It can now be checked that we have the following:

$$
\begin{aligned}
\frac{\partial}{\partial\theta}\, \tilde{Q}_B^{(1c)}(\tau,\theta) = e^2\bigg[&\frac{d}{d\tau}\Big\{ \big[B + \bar{B} + i\big(\bar{C}\dot{C} - \dot{\bar{C}}C\big)\big]\big(i B + \bar{C}\dot{C}\big) \\
&- \frac{d}{d\tau}\big[B + \bar{B} + i\big(\bar{C}\dot{C} - \dot{\bar{C}}C\big)\big]\bar{C}C\Big\} - 2\, i\, \dot{B}\big[B + \bar{B} + i\big(\bar{C}\dot{C} - \dot{\bar{C}}C\big)\big]\bigg] \\
&+ 2\, e\, \dot{e}\, \bar{C}\Big[\big(B + \bar{B} + i\big(\bar{C}\dot{C} - \dot{\bar{C}}C\big)\big) C \\
&- \frac{d}{d\tau}\big[B + \bar{B} + i\big(\bar{C}\dot{C} - \dot{\bar{C}}C\big)\big]C\Big] \Leftrightarrow s_{ab}\, Q_B^{(1)} = -i\{Q_B^{(1)}, Q_{\bar{B}}^{(1)}\},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial\bar{\theta}}\, \tilde{Q}_{\bar{B}}^{(1ac)}(\tau,\bar{\theta}) = 2\, e\, \dot{e}\Big[&\big\{ B + \bar{B} + i\big(\bar{C}\dot{C} - \dot{\bar{C}}C\big)\big\} \dot{\bar{C}}C \\
&- \frac{d}{d\tau}\big\{ B + \bar{B} + i\big(\bar{C}\dot{C} - \dot{\bar{C}}C\big)\big\}\bar{C}C\Big] \\
&+ e^2\bigg[\frac{d}{d\tau}\Big\{ \big[B + \bar{B} + i\big(\bar{C}\dot{C} - \dot{\bar{C}}C\big)\big]\big(\dot{\bar{C}}C - i\bar{B}\big) \\
&- \frac{d}{d\tau}\big[B + \bar{B} + i\big(\bar{C}\dot{C} - \dot{\bar{C}}C\big)\big]\bar{C}C\Big\} \\
&+ 2\, i\, \dot{B}\big[B + \bar{B} + i\big(\bar{C}\dot{C} - \dot{\bar{C}}C\big)\big]\bigg] \Leftrightarrow s_b\, Q_{\bar{B}}^{(1)} = -i\{Q_{\bar{B}}^{(1)}, Q_{\bar{B}}^{(1)}\}.
\end{aligned}
\tag{D.7}
$$

Thus, a careful and close look at the r.h.s. of (D.7) demonstrates that we have deduced the CF-type restriction $B + \bar{B} + i(\bar{C}\dot{C} - \dot{\bar{C}}C) = 0$ within the ambit of ACSA in the proof of the absolute anticommutativity of the conserved (anti-)BRST charges. It is worthwhile to mention here that our Equation (D.7) captures the absolute anticommutativity of the (anti-)BRST charges $[Q_{(\bar{B})B}]$ in the *ordinary* space as well as in the *superspace* provided the whole theory is considered on the subspace (of the entire quantum Hilbert space) of variables where the CF-type restriction $B + \bar{B} + i(\bar{C}\dot{C} - \dot{\bar{C}}C) = 0$ is satisfied.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Acknowledgments

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