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On the Frame Set for the 3-Spline

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Abstract

This paper investigates the fine structure of the Gabor frame generated by the B-spline B_3. In other words, one extends the known part of the Gabor frame set for the 3-spline with the construction of the compactly supported dual windows. The frame set of the function B_3 is the subset of all parameters $(a,b)\in\mathbb{R}_+^2$ for which the time-frequency shifts of B_3 along $a\mathbb{Z}\times b\mathbb{Z}$ form a Gabor frame for $L^2(\mathbb{R})$.

Keywords

Gabor Frames, Frame Set, 3-Splines, Compactly Dual Frame

1. A New Set of Points in the Frame Set of the 3-Spline

The *Gabor frame* generated by $g\in L^2(\mathbb{R})$ and $a,b>0$ is the set of functions

$$\mathcal{G}(g,a,b)=\left\{M_{\ell b}T_{ka}g=\mathrm{e}^{2\pi i\ell b\cdot}g(\cdot-ka):(\ell,k)\in\mathbb{Z}^2\right\}$$

for which there exist $A,B>0$ such that

$$A\|f\|_2^2\le\sum_{\ell,k\in\mathbb{Z}}\left|\left\langle f,M_{\ell b}T_{ka}g\right\rangle\right|^2\le B\|f\|_2^2,$$

for all $f\in L^2(\mathbb{R})$, see, [1] [2] for details. The mystery of the fine structure of Gabor frames [3] and application requirements have obliged many researchers to investigate the characterization of Gabor frame set for some functions having good time-frequency concentration. For example, in the hyperbolic secant functions or different splines windows [4] [5] [6], the functions satisfy the partitions of unity [7] [8] and the sign-changing functions with compact support [9] [10] [11] are already been studied. More generally, the frame set $\mathcal{F}(g)$ of $g\in L^2(\mathbb{R})$ is known only for a few classes of functions, see [3] [12]-[22]. For instance, the determination of the frame set of *B*-splines for $N\ge 2$ is listed as one of the six problems in frame theory [12]. Recently, we have investigated the frame set of a class of compactly supported functions that include the *B*-splines. In particular, we have extended and put in a more general framework some of the known results on the frame set for this class of functions [23].

In this paper, we investigate the set of parameters $(a,b)\in\mathbb{R}_+^2$ such that $\mathcal{G}(B_3,a,b)$ is a Gabor frame where

$$B_3(x)=\chi_{[-1/2,1/2]}*\chi_{[-1/2,1/2]}*\chi_{[-1/2,1/2]}(x)=\begin{cases}\frac{1}{2}x^2+\frac{3}{2}x+\frac{9}{8} & x\in\left[-\frac{3}{2},-\frac{1}{2}\right]\\ -x^2+\frac{3}{4} & x\in\left[-\frac{1}{2},\frac{1}{2}\right]\\ \frac{1}{2}x^2-\frac{3}{2}x+\frac{9}{8} & x\in\left[\frac{1}{2},\frac{3}{2}\right]\end{cases}$$

is the 3-spline. This set is called the frame set of B_3 and is given by

$$\mathcal{F}(B_3)=\left\{(a,b)\in\mathbb{R}_+^2:\mathcal{G}(B_3,a,b)\text{ is a frame}\right\}.$$

It is not difficult to see that $\mathcal{F}(B_3)\neq\varnothing$. Indeed, for any $a\in(0,3)$ there exists $b_0>0$ such that for all $0<b\le b_0$, $(a,b)\in\mathcal{F}(B_3)$ [[24], Theorem 4.18]. Furthermore, $\mathcal{F}(B_3)$ is an open set in $\mathbb{R}_+^2$ [25] [26]. This is a consequence of the fact that B_3 belongs to the modulation space $M^1(\mathbb{R})$ [2]. However, the complete characterization of $\mathcal{F}(B_3)$ is still unknown.

To the best of our knowledge [1] [5] [8] [11] [23] [27] [28], the biggest subset known in $\mathcal{F}(B_3)$ is the connected set

$$\left\{(a,b)\in\mathbb{R}_+^2:ab<1,0<a<3,0<b\le\max_a\left(\frac{2}{3},\frac{4}{3+3a}\right)\right\}.$$

In this note, we complete this last subset by showing that Γ belongs to $\mathcal{F}(B_3)$ where $\Gamma=\bigcup_{m=3}^{\infty}\Gamma_m$ with for $m\ge4$,

$$\Gamma_m:=\left\{(a,b)\in\mathbb{R}_+^2:a\in\left[\frac{3(m-3)}{2(m-2)},\frac{3(m-1)}{2m-1}\right],\right.$$
$$\left.b\in\left(\frac{2(m-1)}{3+(2m-3)a},\min_a\left(\frac{2m}{3+(2m-1)a},\frac{4}{3+2a}\right)\right],b>\frac{2}{3}\right\} \tag{1}$$

and

$$\Gamma_3:=\left\{(a,b)\in\mathbb{R}_+^2:a\in\left[\frac{3}{4},\frac{6}{5}\right],b\in\left(\frac{4}{3+3a},\frac{6}{3+5a}\right],b>\frac{2}{3}\right\}. \tag{2}$$

More specifically, our main result gives the frame properties on Γ.

Theorem 1. *For* $m\ge3$, *let* $(a,b)\in\Gamma_m$. *Then, the Gabor system* $\mathcal{G}(B_3,a,b)$ *is a frame for* $L^2(\mathbb{R})$, *and there is a unique dual window* $H_3\in L^2(\mathbb{R})$ *such that* $\operatorname{supp}H_3\subseteq\left[-\frac{2m-1}{2}a,\frac{2m-1}{2}a\right]$. *Furthermore, for each* $(a,b)\in\Gamma$, *the Gabor system* $\mathcal{G}(B_3,a,b)$ *is a frame for* $L^2(\mathbb{R})$.

Theorem 1 is proved by using the following well-known necessary and sufficient condition, which is well developed in [[23], Proposition 2], for two Bessel

Gabor systems to be dual of each other.

For $m \geq 1$, let $0 < a < 3$ and $\frac{2(m-1)}{3+(2m-3)a} < b \leq \frac{2m}{3+(2m-1)a}$. Assume that H_3 is a bounded real-function with support on $\left[-\frac{2m-1}{2}a, \frac{2m-1}{2}a\right]$. Then, the Gabor systems $\mathcal{G}(B_3, a, b)$ and $\mathcal{G}(H_3, a, b)$ are dual frames for $L^2(\mathbb{R})$ if and only if

$$\sum_{k=1-m}^{m-1} B_3(x - \ell/b + ka) H_3(x + ka) = b\delta_{\ell,0}, |\ell| \leq m-1, \text{ for a.e. } x \in \left[-\frac{a}{2}, \frac{a}{2}\right]. \quad (3)$$

We can rewrite (3) as a matrix-vector equation:

$$\hat{\boldsymbol{E}}_m(x) \hat{\boldsymbol{F}}_m^t(x) = \boldsymbol{B}^t \quad \text{for a.e. } x \in \left[-\frac{a}{2}, \frac{a}{2}\right], \quad (4)$$

where $\hat{\boldsymbol{F}}_m(x)$ and $\boldsymbol{B}$ are the row vectors given respectively by $\hat{\boldsymbol{F}}_m(x) = \left[H_3(x+ka)\right]_{|k| \leq m-1}$ and $\boldsymbol{B} = \left[b\delta_{\ell,0}\right]_{|\ell| \leq m-1}$; $\hat{\boldsymbol{E}}_m(x)$ is the $(2m-1) \times (2m-1)$ matrix-valued function defined by $\hat{\boldsymbol{E}}_m(x) = \left[B_3\left(x - \frac{\ell}{b} + ka\right)\right]_{1-m \leq \ell, k \leq m-1}$. Using the fact that the 3-spline B_3 is supported by $\left[-\frac{3}{2}, \frac{3}{2}\right]$ and $(a,b) \in \Gamma_m$; one observes easily that for all $x \in \left[-\frac{a}{2}, 0\right]$, not only the all $\hat{\boldsymbol{E}}_m(x)$'s on-diagonal entries are positive but also the matrix-valued function $\hat{\boldsymbol{E}}_m(x)$ can be rewritten as the following block matrix:

$$\hat{\boldsymbol{E}}_m(x) = \begin{pmatrix} \hat{\boldsymbol{G}}_{m-1}(x) & \hat{\boldsymbol{I}}_{m-1}(x) \\ \boldsymbol{O} & \hat{\boldsymbol{J}}_{m-1}(x) \end{pmatrix}, \quad (5)$$

where $\boldsymbol{O}$ is a $(m-2) \times (m+1)$ matrix of 0's, $\hat{\boldsymbol{I}}_{m-1}(x)$ is a $(m+1) \times (m-2)$ matrix, $\hat{\boldsymbol{J}}_{m-1}(x)$ is a $(m-2) \times (m-2)$ upper triangular matrix and $\hat{\boldsymbol{G}}_{m-1}(x)$ is the $(m+1) \times (m+1)$ tridiagonal matrix given by

$$\hat{\boldsymbol{G}}_{m-1}(x) = \left[B_3\left(x - \frac{\ell}{b} + ka\right)\right]_{1-m \leq \ell, k \leq 1}. \quad (6)$$

Figure 1 shows the connected set known in $\mathcal{F}(B_3)$.

The rest of the paper is organized as follows. In Section 2 we prove our main results by studying the invertibility of the matrix $\hat{\boldsymbol{G}}_{m-1}(x)$.

2. Invertibility of $\hat{G}_{m-1}(x)$ for $(a,b) \in \Gamma_m$

To prove Theorem 1, we only need to show that (3) has a unique solution $\hat{\boldsymbol{F}}_m(x)$. This is equivalent to proving respectively that the $(2m-1) \times (2m-1)$ matrix-valued function $\hat{\boldsymbol{E}}_m(x)$ is invertible for a.e. $x \in \left[-\frac{a}{2}, \frac{a}{2}\right]$. In other

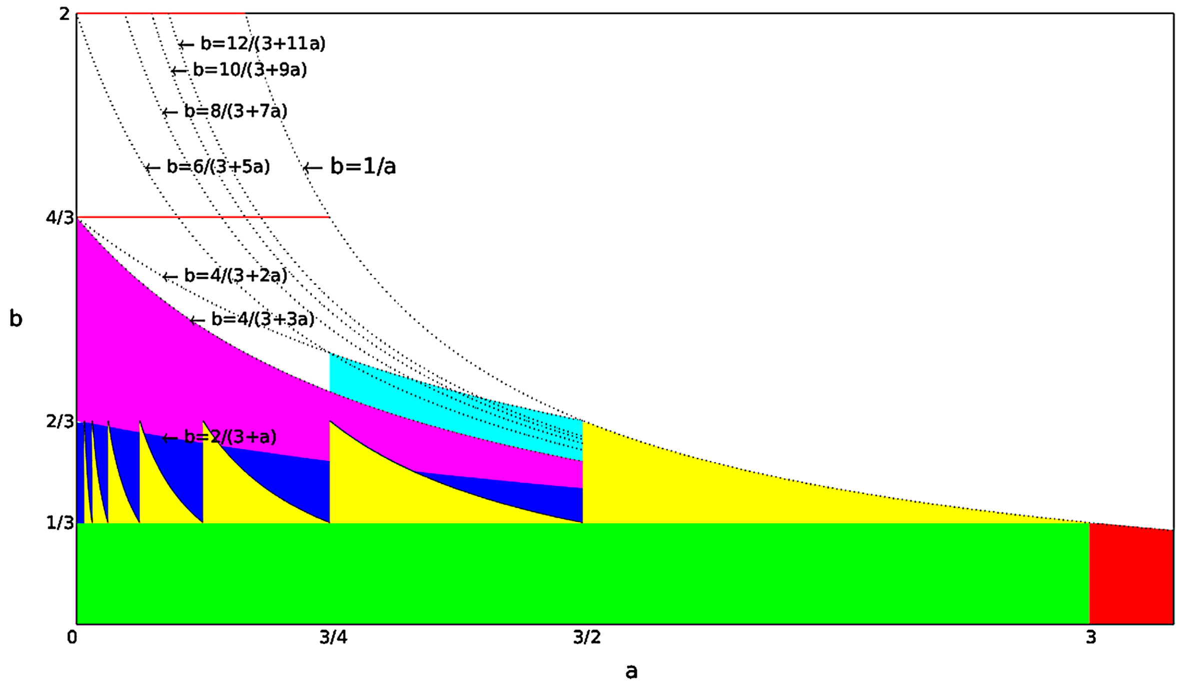

Figure 1. A sketch of $\mathcal{F}(B_3)$. No frame property in the red region, the green region is from [29] and the yellow region is the result from [11]. The blue and the magenta regions are respectively from [28] and [27]. The cyan region is the result from [23] completed by theorem 1.

words, from the block form for the matrix $\hat{\boldsymbol{E}}_m(x)$, we need to prove that the matrix $\hat{\boldsymbol{G}}_{m-1}(x)$ is invertible.

Throughout the paper, we use the fact that

$$\min_a\left(\frac{2m}{3+(2m-1)a},\frac{4}{3+2a}\right)=\begin{cases}\dfrac{4}{3+2a} & \text{if } a\in\left[\dfrac{3(m-3)}{2(m-2)},\dfrac{3(m-2)}{2(m-1)}\right]\\[2ex] \dfrac{2m}{3+(2m-1)a} & \text{if } a\in\left[\dfrac{3(m-2)}{2(m-1)},\dfrac{3(m-1)}{2m-1}\right]\end{cases}$$

The following lemma gives the behavior of the entries of $\hat{\boldsymbol{G}}_{m-1}(x)$.

Lemma 1. *For* $m\geq 3$*, let* $(a,b)\in\Gamma_m$*,* $x\in\left[-\frac{a}{2};0\right]$ *and consider* B_3*. Then the following hold*:

1) $B_3\left(x+\frac{k}{b}-ka\right)>0$, for all $k\in\{1-m,\cdots,m-1\}$.

2) $B_3\left(x+\frac{k}{b}-(k-1)a\right)>0$ for $k\in\{0;1\}$ and

$$B_3\left(x+\frac{k}{b}-(k-1)a\right)=\begin{cases}B_3\left(x+\frac{k}{b}-(k-1)a\right)>0 & \text{if } x\in\left[-\frac{a}{2};\frac{3}{2}-\frac{k}{b}+(k-1)a\right)\\[1ex] 0 & \text{if } x\in\left[\frac{3}{2}-\frac{k}{b}+(k-1)a;0\right]\end{cases}$$

for all $k \in \{2,\cdots,m-1\}$.

3) $B_3\left(x+\frac{m-2}{b}-(m-1)a\right)>0$ and

$$B_3\left(x+\frac{k}{b}-(k+1)a\right)$$
$$=\begin{cases}0 & \text{if } x\in\left[-\frac{a}{2};-\frac{3}{2}-\frac{k}{b}+(k+1)a\right]\\ B_3\left(x+\frac{k}{b}-(k+1)a\right)>0 & \text{if } x\in\left(-\frac{3}{2}-\frac{k}{b}+(k+1)a;0\right]\end{cases}$$

for all $\{k=-1,\cdots,m-3\}$.

Proof. 1) Firstly, let $k\in\{1-m,\cdots,-1\}$. We have

$$x+\frac{k}{b}-ka\le\frac{k}{b}-ka$$
$$\le\begin{cases}k\left(\frac{3+2a}{4}\right)-ka=k\left(\frac{3-2a}{4}\right) & \text{if } a\in\left[\frac{3(m-3)}{2(m-2)},\frac{3(m-2)}{2(m-1)}\right]\\ k\left(\frac{3+(2m-1)a}{2m}\right)-ka=\frac{3k}{2m}-\frac{k}{2m}a & \text{if } a\in\left[\frac{3(m-2)}{2(m-1)},\frac{3(m-1)}{2m-1}\right]\end{cases}$$
$$\le\begin{cases}\frac{3k}{4}-\frac{k}{2}\times\frac{3(m-2)}{2(m-1)}=\frac{3k}{4(m-1)} & \text{if } a\in\left[\frac{3(m-3)}{2(m-2)},\frac{3(m-2)}{2(m-1)}\right]\\ \frac{3k}{2m}-\frac{k}{2m}\times\frac{3(m-1)}{2m-1}=\frac{3k}{2(2m-1)} & \text{if } a\in\left[\frac{3(m-2)}{2(m-1)},\frac{3(m-1)}{2m-1}\right]\end{cases}$$
$$<0$$

and

$$x+\frac{k}{b}-ka\ge\frac{k}{b}-\frac{2k+1}{2}a$$
$$>k\left(\frac{3+(2m-3)a}{2(m-1)}\right)-\frac{2k+1}{2}a=\frac{3k}{2(m-1)}-\frac{m-1+k}{2(m-1)}a$$
$$\ge\frac{3k}{2(m-1)}-\frac{m-1+k}{2(m-1)}\times\frac{3(m-1)}{2m-1}=\frac{3}{2}\left(\frac{k}{m-1}-\frac{m-1+k}{2m-1}\right)$$
$$\ge-\frac{3}{2}\quad\left(\text{because } \frac{k}{m-1}-\frac{m-1+k}{2m-1}\ge-1\right).$$

Therefore for all $k\in\{1-m,\cdots,-1\}$, $-\frac{3}{2}<x+\frac{k}{b}-ka<0$. Thus $B_3\left(x+\frac{k}{b}-ka\right)>0$. Secondly, let $k\in\{0,\cdots,m-1\}$. We have

$$x+\frac{k}{b}-ka\le\frac{k}{b}-ka<k\left(\frac{3+(2m-3)a}{2(m-1)}\right)-ka=\frac{3k}{2(m-1)}-\frac{ka}{2(m-1)}$$
$$\le\frac{3k}{2(m-1)}-\frac{k}{2(m-1)}\times\frac{3(m-3)}{2(m-2)}=\frac{k}{4(m-2)}\le\frac{3}{2}$$

and

$$x+\frac{k}{b}-ka \geq -\frac{a}{2}+\frac{k}{b}-ka=\frac{k}{b}-\frac{2k+1}{2}a$$

$$\geq \begin{cases} k\left(\frac{3+2a}{4}\right)-\frac{2k+1}{2}a=\frac{3k}{4}-\frac{k+1}{2}a & \text{if } a\in\left[\frac{3(m-3)}{2(m-2)},\frac{3(m-2)}{2(m-1)}\right] \\ k\left(\frac{3+(2m-1)a}{2m}\right)-\frac{2k+1}{2}a=\frac{3k}{2m}-\frac{m+k}{2m}a & \text{if } a\in\left[\frac{3(m-2)}{2(m-1)},\frac{3(m-1)}{2m-1}\right] \end{cases}$$

$$\geq \begin{cases} \frac{3k}{4}-\frac{k+1}{2}\times\frac{3(m-2)}{2(m-1)}=-\frac{3}{2}\left(\frac{(m-2)-k}{2(m-1)}\right) \\ \frac{3k}{2m}-\frac{m+k}{2m}\times\frac{3(m-1)}{2m-1}=-\frac{3}{2}\left(\frac{(m-1)-k}{2m-1}\right) \end{cases}$$

$$>-\frac{3}{2}$$

Then for all $k\in\{0,\cdots,m-1\}$, $-\frac{3}{2}<x+\frac{k}{b}-ka<\frac{3}{2}$. Thus $B_3\left(x+\frac{k}{b}-ka\right)>0$.

2) Let $k\in\{0,\cdots,m-1\}$. We have

$$\begin{aligned} x+\frac{k}{b}-(k-1)a &\leq \frac{k}{b}-(k-1)a<k\left(\frac{3+(2m-3)a}{2(m-1)}\right)-(k-1)a \\ &=\frac{3k}{2(m-1)}+\frac{2(m-1)-k}{2(m-1)}a \\ &\leq \frac{3k}{2(m-1)}+\frac{2(m-1)-k}{2(m-1)}\times\frac{3(m-1)}{2m-1} \\ &=\frac{3}{2}\left(\frac{k}{m-1}+\frac{2(m-1)-k}{2m-1}\right) \end{aligned}$$

and

$$x+\frac{k}{b}-(k-1)a \geq -\frac{a}{2}+\frac{k}{b}-(k-1)a=\frac{k}{b}-\frac{2k-1}{2}a$$

$$\geq \begin{cases} k\left(\frac{3+2a}{4}\right)-\frac{2k-1}{2}a=\frac{3k}{4}-\frac{k-1}{2}a & \text{if } a\in\left[\frac{3(m-3)}{2(m-2)},\frac{3(m-2)}{2(m-1)}\right] \\ k\left(\frac{3+(2m-1)a}{2m}\right)-\frac{2k-1}{2}a=\frac{3k}{2m}+\frac{m-k}{2m}a & \text{if } a\in\left[\frac{3(m-2)}{2(m-1)},\frac{3(m-1)}{2m-1}\right] \end{cases}$$

$$\geq \begin{cases} \frac{3k}{4}-\frac{k-1}{2}\times\frac{3(m-2)}{2(m-1)}=\frac{3}{2}\left(\frac{m+k-2}{2(m-1)}\right) & \text{if } a\in\left[\frac{3(m-3)}{2(m-2)},\frac{3(m-2)}{2(m-1)}\right] \\ \frac{3k}{2m}+\frac{m-k}{2m}\times\frac{3(m-2)}{2(m-1)}=\frac{3}{2}\left(\frac{m+k-2}{2(m-1)}\right) & \text{if } a\in\left[\frac{3(m-2)}{2(m-1)},\frac{3(m-1)}{2m-1}\right] \end{cases}$$

Then for all $k\in\{1,\cdots,m-1\}$;

$$0<\frac{3}{2}\left(\frac{m+k-2}{2(m-1)}\right)\leq x+\frac{k}{b}-(k-1)a<\frac{3}{2}\left(\frac{k}{m-1}+\frac{2(m-1)-k}{2m-1}\right).$$

Therefore for all $k\in\{2,\cdots,m-1\}$,

$$B_3\left(x+\frac{k}{b}-(k-1)a\right)=\begin{cases}B_3\left(x+\frac{k}{b}-(k-1)a\right)>0 & \text{if } x\in\left[-\frac{a}{2};\frac{3}{2}-\frac{k}{b}+(k-1)a\right)\\ 0 & \text{if } x\in\left[\frac{3}{2}-\frac{k}{b}+(k-1)a;0\right]\end{cases}$$

and using the fact that $b>\frac{2}{3}$, we have easily $B_3\left(x+\frac{k}{b}-(k-1)a\right)>0$ for $k\in\{0;1\}$.

3) Let $k\in\{0,\cdots,m-2\}$. We have

$$\begin{aligned}x+\frac{k}{b}-(k+1)a\le\frac{k}{b}-(k+1)a&<k\left(\frac{3+(2m-3)a}{2(m-1)}\right)-(k+1)a\\&=\frac{3k}{2(m-1)}-\frac{2(m-1)+k}{2(m-1)}a\\&\le\frac{3k}{2(m-1)}-\frac{2(m-1)+k}{2(m-1)}\times\frac{3(m-3)}{2(m-2)}\\&=\frac{3}{2}\left(\frac{k-2m+6}{2(m-2)}\right)=\frac{3}{2}\left(\frac{(k-m+2)+(4-m)}{2(m-2)}\right)\le 0\end{aligned}$$

as far as $m\ge 4$ and

$$x+\frac{k}{b}-(k+1)a\ge-\frac{a}{2}+\frac{k}{b}-(k+1)a=\frac{k}{b}-\frac{2k+3}{2}a$$

$$\ge\begin{cases}k\left(\frac{3+2a}{4}\right)-\frac{2k+3}{2}a=\frac{3k}{4}-\frac{k+3}{2}a & \text{if } a\in\left[\frac{3(m-3)}{2(m-2)},\frac{3(m-2)}{2(m-1)}\right]\\ k\left(\frac{3+(2m-1)a}{2m}\right)-\frac{2k+3}{2}a=\frac{3k}{2m}-\frac{3(m-1)+k+3}{2m}a & \text{if } a\in\left[\frac{3(m-2)}{2(m-1)},\frac{3(m-1)}{2m-1}\right]\end{cases}$$

$$\ge\begin{cases}\frac{3k}{4}-\frac{k+3}{2}\times\frac{3(m-2)}{2(m-1)}=\frac{3}{2}\left(\frac{k-3(m-2)}{2(m-1)}\right) & \text{if } a\in\left[\frac{3(m-3)}{2(m-2)},\frac{3(m-2)}{2(m-1)}\right]\\ \frac{3k}{2m}-\frac{3(m-1)+k+3}{2m}\times\frac{3(m-1)}{2m-1}=\frac{3}{2}\left(\frac{k-3(m-1)}{2m-1}\right) & \text{if } a\in\left[\frac{3(m-2)}{2(m-1)},\frac{3(m-1)}{2m-1}\right]\end{cases}$$

Then for all $k\in\{0,\cdots,m-2\}$;

$$\begin{cases}\frac{3}{2}\left(\frac{k-3(m-2)}{2(m-1)}\right)\\ \frac{3}{2}\left(\frac{k-3(m-1)}{2m-1}\right)\end{cases}\le x+\frac{k}{b}-(k+1)a<\frac{3}{2}\left(\frac{(k-m+2)+(4-m)}{2(m-2)}\right)\le 0$$

as far as $m\ge 4$ and

$$-\frac{3}{2}-\frac{3\left((m-1)^2+1\right)}{2(m-1)(2m-1)}<x-\frac{1}{b}\le\begin{cases}-\frac{3(2m-5)}{4(m-2)} & \text{if } a\in\left[\frac{3(m-3)}{2(m-2)},\frac{3(m-2)}{2(m-1)}\right]\\ -\frac{3(2m-3)}{4(m-1)} & \text{if } a\in\left[\frac{3(m-2)}{2(m-1)},\frac{3(m-1)}{2m-1}\right]\end{cases}<0.$$

Thus $B_3\left(x+\frac{m-2}{b}-(m-1)a\right)>0$ and

$$B_3\left(x+\frac{k}{b}-(k+1)a\right) = \begin{cases} 0 & \text{if } x\in\left[-\frac{a}{2};-\frac{3}{2}-\frac{k}{b}+(k+1)a\right] \\ B_3\left(x+\frac{k}{b}-(k+1)a\right)>0 & \text{if } x\in\left[-\frac{3}{2}-\frac{k}{b}+(k+1)a;0\right] \end{cases}$$

for $k=-1,\cdots,m-3$.

The following remark gives the trivial two different partitions of $\left[-\frac{a}{2};0\right]$ which can be derived from the definition of $\Gamma_m(m\ge 3)$ and will be used to compute the determinant of the matrix $\hat{\boldsymbol{G}}_{m-1}(x)$.

Remark 1. a) If $3-\frac{3}{b}+a\le 0$ and

- $a\in\left[\frac{3(m-3)}{2(m-2)},\frac{6m-9}{4m-3}\right]$, then $-\frac{3}{2}-\frac{m-3}{b}+(m-2)a\le-\frac{a}{2}$. Hence

$$\left[-\frac{a}{2},0\right]=\bigcup_{k=1}^{m-2}(Q_k\cup T_k)\cup T_{m-1}$$

where $Q_k=\left[\frac{3}{2}-\frac{k+1}{b}+ka,-\frac{3}{2}-\frac{k-2}{b}+(k-1)a\right]$ and $T_k=\left(-\frac{3}{2}-\frac{k-2}{b}+(k-1)a,\frac{3}{2}-\frac{k}{b}+(k-1)a\right)$ with the convention that $T_{m-1}=\left[-\frac{a}{2},\frac{3}{2}-\frac{m-1}{b}+(m-2)a\right)$ and $T_1=\left(-\frac{3}{2}+\frac{1}{b},0\right]$.

- $a\in\left[\frac{3(m-2)}{2m-3},\frac{3(m-1)}{2m-1}\right]$, then $-\frac{a}{2}\le-\frac{3}{2}-\frac{m-3}{b}+(m-2)a$. Thus

$$\left[-\frac{a}{2},0\right]=\bigcup_{k=1}^{m-1}(Q_k\cup T_k)$$

where $Q_k=\left[\frac{3}{2}-\frac{k+1}{b}+ka,-\frac{3}{2}-\frac{k-2}{b}+(k-1)a\right]$ and $T_k=\left(-\frac{3}{2}-\frac{k-2}{b}+(k-1)a,\frac{3}{2}-\frac{k}{b}+(k-1)a\right)$ with the convention that $Q_{m-1}=\left[-\frac{a}{2},-\frac{3}{2}-\frac{m-3}{b}+(m-2)a\right]$ and $T_1=\left(-\frac{3}{2}+\frac{1}{b},0\right]$.

- $a\in\left(\frac{6m-9}{4m-3};\frac{3(m-2)}{2m-3}\right)$ for $m\ne 3$, then $-\frac{3}{2}-\frac{m-3}{b}+(m-2)a+\frac{a}{2}$ is sign-changing. So, we use both different previous partitions.

b) If $3-\frac{3}{b}+a>0$, then a is only in $\left[\frac{3(m-3)}{2(m-2)},\frac{6m-9}{4m-3}\right]$. Hence

$$\left[-\frac{a}{2},0\right]=\left(\bigcup_{k=1}^{m-1}\tilde{Q}_k\right)\cup\left(\bigcup_{k=2}^{m-1}\tilde{T}_k\right)$$

where $\tilde{Q}_k=\left[\frac{3}{2}-\frac{k+1}{b}+ka,-\frac{3}{2}-\frac{k-3}{b}+(k-2)a\right]$ and

$\tilde{T}_k=\left(-\frac{3}{2}-\frac{k-3}{b}+(k-2)a,\frac{3}{2}-\frac{k}{b}+(k-1)a\right)$ with the convention that

$\tilde{Q}_{m-1}=\left[-\frac{a}{2},-\frac{3}{2}-\frac{m-4}{b}+(m-3)a\right]$ and $\tilde{Q}_1=\left[\frac{3}{2}-\frac{2}{b}+a,0\right]$.

NB: Specially for $m=3$, one substitutes $\frac{3(m-3)}{2(m-2)}$ by $\frac{3}{4}$. The different cases considered in proving this result are illustrated for the cases $m=3$ and $m=4$ in Figure 2.

Let $k\in\{1,\cdots,m-1\}$, $\ell=2,3,4$ and denote $A_{\ell\ell}^k(x)$ the $\ell\times\ell$ sub-matrix of $\hat{G}_{m-1}(x)$, defined respectively by

$$A_{22}^k(x)=\begin{pmatrix} B_3\left(x+\frac{k}{b}-ka\right) & B_3\left(x+\frac{k}{b}-(k-1)a\right) \\ B_3\left(x+\frac{k-1}{b}-ka\right) & B_3\left(x+\frac{k-1}{b}-(k-1)a\right)\end{pmatrix},$$

$$A_{33}^k(x)=\begin{pmatrix} B_3\left(x+\frac{k}{b}-ka\right) & B_3\left(x+\frac{k}{b}-(k-1)a\right) & 0 \\ B_3\left(x+\frac{k-1}{b}-ka\right) & B_3\left(x+\frac{k-1}{b}-(k-1)a\right) & B_3\left(x+\frac{k-1}{b}-(k-2)a\right) \\ 0 & B_3\left(x+\frac{k-2}{b}-(k-1)a\right) & B_3\left(x+\frac{k-2}{b}-(k-2)a\right)\end{pmatrix}$$

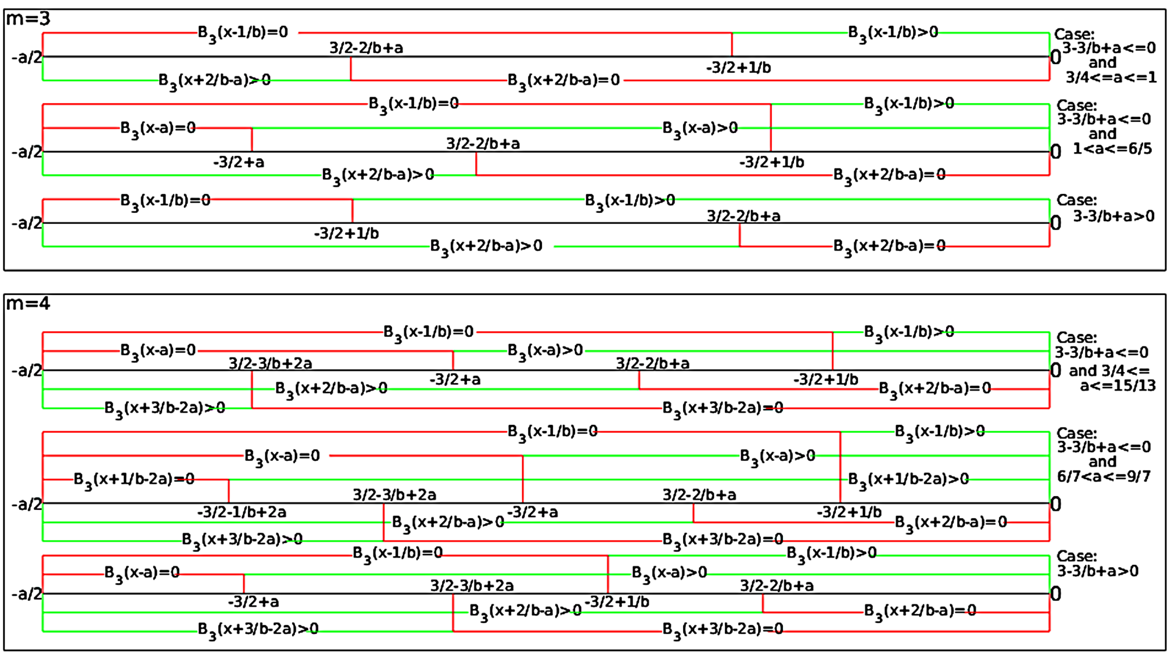

Figure 2. The off-diagonal of $\hat{G}_{m-1}(x)$ for $m=3$ and $m=4$ when $x\in\left[-\frac{a}{2},0\right]$.

and

$$A_{44}^{k}(x)=\begin{pmatrix} B_3\left(x+\frac{k}{b}-ka\right) & B_3\left(x+\frac{k}{b}-(k-1)a\right) & 0 & 0 \\ B_3\left(x+\frac{k-1}{b}-ka\right) & B_3\left(x+\frac{k-1}{b}-(k-1)a\right) & B_3\left(x+\frac{k-1}{b}-(k-2)a\right) & 0 \\ 0 & B_3\left(x+\frac{k-2}{b}-(k-1)a\right) & B_3\left(x+\frac{k-2}{b}-(k-2)a\right) & B_3\left(x+\frac{k-2}{b}-(k-3)a\right) \\ 0 & 0 & B_3\left(x+\frac{k-3}{b}-(k-2)a\right) & B_3\left(x+\frac{k-3}{b}-(k-3)a\right) \end{pmatrix}.$$

The following Lemma indicates that the matrices $A_{\ell\ell}^{k}(x)$ are invertible.

Lemma 2. *Given* $m\ge 3$, *let* $(a,b)\in\Gamma_m$ *and* $x\in\left[-\frac{a}{2};0\right]$. *The following hold*:

a) if $k\in\{1,\cdots,m-1\}$, then $\left|A_{\ell\ell}^{k}(x)\right|>0$ for all $\ell=2,3$.

b) $\left|A_{44}^{k}(x)\right|>0$, for all $k\in\{2,\cdots,m-1\}$.

This result is proved in Appendix 4.

The following Proposition gives an explicit expression for the determinant of the sub-matrix $\hat{G}_{m-1}(x)$ when $m\ge 3$, $x\in\left[-\frac{a}{2};0\right]$ and under the hypotheses of Theorem 1.

Proposition 1. *For* $m\ge 3$, *let* $(a,b)\in\Gamma_m$ *and* $x\in\left[-\frac{a}{2},0\right]$. *The following statements hold*:

$$\forall\, x\in D, \left|\hat{G}_{m-1}(x)\right|=\prod_{l\in I}B_3\left(x+\frac{\ell}{b}-\ell a\right)\cdot\left|A_{jj}^{i}(x)\right|$$

where D, I, i and j *are given in the following cases*:

1) If $3-\frac{3}{b}+a\le 0$ and

a) $\frac{3(m-3)}{2(m-2)}\le a\le\frac{6m-9}{4m-3}$, then

i) for all $k\in\{1,\cdots,m-2\}$, $D=Q_k$, $I=\{-1,\cdots,m-1\}\setminus\{k,k-1\}$, $i=k$, and $j=2$.

ii) for all $k\in\{1,\cdots,m-1\}$, $D=T_k$, $I=\{-1,\cdots,m-1\}\setminus\{k,k-1,k-2\}$, $i=k$ and $j=3$.

b) $\frac{3(m-2)}{2m-3}\le a\le\frac{3(m-1)}{2m-1}$, then for all $k\in\{1,\cdots,m-1\}$, one has:

i) for $D=Q_k$, $I=\{-1,\cdots,m-1\}\setminus\{k,k-1\}$, $i=k$, and $j=2$.

ii) for $D=T_k$, $I=\{-1,\cdots,m-1\}\setminus\{k,k-1,k-2\}$, $i=k$ and $j=3$.

c) $\frac{6m-9}{4m-3}<a<\frac{3(m-2)}{2m-3}$ when $m\ne 3$, then one uses i) and ii) obtained in a) or b).

2) If $3-\frac{3}{b}+a>0$, then

a) For all $k \in \{1,\cdots,m-1\}$, $D = \tilde{Q}_k$, $I = \{-1,\cdots,m-1\} \setminus \{k,k-1,k-2\}$, $i = k$ and $j = 3$.

b) For all $k \in \{2,\cdots,m-1\}$, $D = \tilde{T}_k$, $I = \{-1,\cdots,m-1\} \setminus \{k,k-1,k-2,k-3\}$, $i = k$ and $j = 4$.

Moreover the matrix $\hat{\boldsymbol{E}}_m(x)$ is invertible.

Proof. We prove the result by induction on *m* by using the different partitions of $\left[-\frac{a}{2};0\right]$ obtained in Remark 1. For $m = 3$, the matrix $\hat{\boldsymbol{G}}_2(x)$ given by

$$\hat{\boldsymbol{G}}_2(x) = \begin{pmatrix} B_3\left(x+\frac{2}{b}-2a\right) & B_3\left(x+\frac{2}{b}-a\right) & 0 & 0 \\ B_3\left(x+\frac{1}{b}-2a\right) & B_3\left(x+\frac{1}{b}-a\right) & B_3\left(x+\frac{1}{b}\right) & 0 \\ 0 & B_3(x-a) & B_3(x) & B_3(x+a) \\ 0 & 0 & B_3\left(x-\frac{1}{b}\right) & B_3\left(x-\frac{1}{b}+a\right) \end{pmatrix}.$$

Suppose that $3-\frac{3}{b}+a \le 0$ and $\frac{3}{4} \le a \le 1$, then $\left[-\frac{a}{2};0\right] = Q_1 \cup T_1 \cup T_2$ with $T_2 = \left[-\frac{a}{2};\frac{3}{2}-\frac{2}{b}+a\right)$, $Q_1 = \left[\frac{3}{2}-\frac{2}{b}+a;-\frac{3}{2}+\frac{1}{b}\right]$ and $T_1 = \left(-\frac{3}{2}+\frac{1}{b};0\right]$.

So, it is easy to see that for all $x \in Q_1$,

$\left|\hat{\boldsymbol{G}}_2(x)\right| = B_3\left(x+\frac{2}{b}-2a\right)B_3\left(x-\frac{1}{b}+a\right)\cdot\left|\boldsymbol{A}_{22}^1(x)\right|$; $x \in T_1$,

$\left|\hat{\boldsymbol{G}}_2(x)\right| = B_3\left(x+\frac{2}{b}-2a\right)\cdot\left|\boldsymbol{A}_{33}^1(x)\right|$ and $x \in T_2$,

$\left|\hat{\boldsymbol{G}}_2(x)\right| = B_3\left(x-\frac{1}{b}+a\right)\cdot\left|\boldsymbol{A}_{33}^2(x)\right|$. This establishes the part (a) for the base case $m = 3$.

Suppose that $3-\frac{3}{b}+a \le 0$ and $1 < a \le \frac{6}{5}$, then $\left[-\frac{a}{2};0\right] = Q_2 \cup T_2 \cup Q_1 \cup T_1$ with $Q_2 = \left[-\frac{a}{2};-\frac{3}{2}+a\right]$, $T_2 = \left(-\frac{3}{2}+a;\frac{3}{2}-\frac{2}{b}+a\right)$, $Q_1 = \left[\frac{3}{2}-\frac{2}{b}+a;-\frac{3}{2}+\frac{1}{b}\right]$ and $T_1 = \left(-\frac{3}{2}+\frac{1}{b};0\right]$. So, it is easy to see that

$$\forall x \in D = Q_k\ (k=1,2), \left|\hat{\boldsymbol{G}}_2(x)\right| = \prod_{l\in I} B_3\left(x+\frac{\ell}{b}-\ell a\right)\cdot\left|\boldsymbol{A}_{jj}^i(x)\right|$$

where $I = \{-1,0,1,2\} \setminus \{k,k-1\}$, $i = k$, and $j = 2$; and

$$\forall x \in D = T_k\ (k=1,2), \left|\hat{\boldsymbol{G}}_2(x)\right| = \prod_{l\in I} B_3\left(x+\frac{\ell}{b}-\ell a\right)\cdot\left|\boldsymbol{A}_{jj}^i(x)\right|$$

where $I = \{-1,0,1,2\} \setminus \{k,k-1,k-2\}$, $i = k$ and $j = 3$. Hence (b) holds. Similarly, (2) holds.

Suppose that (1) and (2) hold for $m-1 \ge 3$ and let us prove that they hold for *m*. So, one compute the determinant of the matrix $\hat{\boldsymbol{G}}_m(x)$ given by

$$\hat{\boldsymbol{G}}_m(x)=\left[B_3\left(x-\frac{\ell}{b}+ka\right)\right]_{-m\le\ell,k\le1}.$$

Suppose $3-\frac{3}{b}+a>0$. From Remark 1, we have

$\left[-\frac{a}{2};0\right]=\left(\bigcup_{k=1}^{m}\tilde{Q}_k\right)\cup\left(\bigcup_{k=2}^{m}\tilde{T}_k\right)$ with $\tilde{Q}_m=\left[-\frac{a}{2};-\frac{3}{2}-\frac{m-3}{b}+(m-2)a\right]$;

$\tilde{Q}_k=\left[\frac{3}{2}-\frac{k+1}{b}+ka;-\frac{3}{2}-\frac{k-3}{b}+(k-2)a\right]$, for all $k\in\{2,\cdots,m-1\}$;

$\tilde{Q}_1=\left[\frac{3}{2}-\frac{2}{b}+a;0\right]$ and $\tilde{T}_k=\left(-\frac{3}{2}-\frac{k-3}{b}+(k-2)a;\frac{3}{2}-\frac{k}{b}+(k-1)a\right)$, for all $k\in\{2,\cdots,m\}$.

Let $x\in D=\tilde{Q}_m$. Thus $B_3\left(x+\frac{k}{b}-(k+1)a\right)=0$ for all $k\in\{-1,\cdots,m-3\}$. Hence,

$$\left|\hat{\boldsymbol{G}}_m(x)\right|=\left(\prod_{\ell=-1}^{m-3}B_3\left(x+\frac{\ell}{b}-\ell a\right)\right)\cdot\left|\boldsymbol{A}_{33}^{m}(x)\right|=\prod_{l\in I}B_3\left(x+\frac{\ell}{b}-\ell a\right)\cdot\left|\boldsymbol{A}_{jj}^{i}(x)\right|$$

where $I=\{-1,\cdots,m-3\}$, $i=m$ and $j=3$.

Let $x\in D=\tilde{Q}_{m-1}$. Therefore $B_3\left(x+\frac{m}{b}-(m-1)a\right)=0$ and $B_3\left(x+\frac{k}{b}-(k+1)a\right)=0$ for all $k\in\{-1,\cdots,m-4\}$. Consequently,

$$\left|\hat{\boldsymbol{G}}_m(x)\right|=\left(\prod_{\substack{\ell=-1\\ \ell\ne m-3,m-2,m-1}}^{m}B_3\left(x+\frac{\ell}{b}-\ell a\right)\right)\cdot\left|\boldsymbol{A}_{33}^{m-1}(x)\right|=\prod_{l\in I}B_3\left(x+\frac{\ell}{b}-\ell a\right)\cdot\left|\boldsymbol{A}_{jj}^{i}(x)\right|$$

where $I=\{-1,\cdots,m\}\setminus\{m-3,m-2,m-1\}$, $i=m-1$ and $j=3$.

Let $k\in\{1,\cdots,m-2\}$ and $x\in D=\tilde{Q}_k$. Thus $B_2\left(x+\frac{m}{b}-(m-1)a\right)=0$. Hence, $\left|\hat{\boldsymbol{G}}_m(x)\right|=B_3\left(x+\frac{m}{b}-ma\right)\cdot\left|\hat{\boldsymbol{G}}_{m-1}(x)\right|$ and by the induction assumption, we have

$$\left|\hat{\boldsymbol{G}}_m(x)\right|=\left(\prod_{\substack{\ell=-1\\ \ell\ne k,k-1,k-2}}^{m}B_3\left(x+\frac{\ell}{b}-\ell a\right)\right)\cdot\left|\boldsymbol{A}_{33}^{k}(x)\right|=\prod_{l\in I}B_3\left(x+\frac{\ell}{b}-\ell a\right)\cdot\left|\boldsymbol{A}_{jj}^{i}(x)\right|$$

where $I=\{-1,\cdots,m\}\setminus\{k,k-1,k-2\}$, $i=k$ and $j=3$.

Let $x\in D=\tilde{T}_m$. Thus $B_3\left(x+\frac{k}{b}-(k+1)a\right)=0$ for all $k\in\{-1,\cdots,m-4\}$. Hence,

$$\left|\hat{\boldsymbol{G}}_m(x)\right|=\left(\prod_{\substack{\ell=-1\\ \ell\ne m-3,m-2,m-1,m}}^{m}B_3\left(x+\frac{\ell}{b}-\ell a\right)\right)\cdot\left|\boldsymbol{A}_{44}^{m}(x)\right|=\prod_{l\in I}B_3\left(x+\frac{\ell}{b}-\ell a\right)\cdot\left|\boldsymbol{A}_{jj}^{i}(x)\right|$$

where $I=\{-1,\cdots,m\}\setminus\{m-3;m-2,m-1,m\}$, $i=m$ and $j=4$.

Let $k \in \{2,\cdots,m-1\}$ and $x \in D = \tilde{T}_k$. Then $B_3\left(x+\frac{m}{b}-(m-1)a\right)=0$. Thus $\left|\hat{\boldsymbol{G}}_m(x)\right| = B_3\left(x+\frac{m}{b}-ma\right)\cdot\left|\hat{\boldsymbol{G}}_{m-1}(x)\right|$ and by the induction assumption, we have

$$\left|\hat{\boldsymbol{G}}_m(x)\right| = \left(\prod_{\substack{\ell=-1\\ \ell\neq k,k-1,k-2,k-3}}^{m} B_3\left(x+\frac{\ell}{b}-\ell a\right)\right)\cdot\left|\boldsymbol{A}_{44}^{k}(x)\right| = \prod_{l\in I} B_3\left(x+\frac{\ell}{b}-\ell a\right)\cdot\left|\boldsymbol{A}_{jj}^{i}(x)\right|.$$

where $I=\{-1,\cdots,m\}\setminus\{k,k-1,k-2,k-3\}$, $i=k$ and $j=4$.

Together, (2) holds for the case *m*. Similarly, one proves that (1) holds for the case *m*.

To end the proof, we observe that by using the block decomposition of $\hat{\boldsymbol{E}}_m(x)$, we have for all $x\in\left[-\frac{a}{2};0\right]$,

$$\left|\hat{\boldsymbol{E}}_m(x)\right| = \left(\prod_{1-m}^{-2} B_3\left(x+\frac{k}{b}-ka\right)\right)\cdot\left|\hat{\boldsymbol{G}}_{m-1}(x)\right|.$$

We know that $B_3\left(x+\frac{k}{b}-ka\right)>0$, $\forall\, k\in\{1-m,\cdots,m-1\}$ and $\left|\hat{\boldsymbol{G}}_{m-1}(x)\right|>0$ because $\left|\boldsymbol{A}_{\ell\ell}^{k}(x)\right|>0$ from Lemma 2. Thus we conclude that $\left|\hat{\boldsymbol{E}}_m(x)\right|>0$ for all $x\in\left[-\frac{a}{2};0\right]$, and by symmetry $\left(\left|\hat{\boldsymbol{E}}_m(-x)\right|=\left|\hat{\boldsymbol{E}}_m(x)\right|\right)$ this holds for all $x\in\left[-\frac{a}{2};\frac{a}{2}\right]$.

We are now ready to prove Theorem 1.

Proof of Theorem 1. By Proposition 1 we know that $\hat{\boldsymbol{E}}_m(x)$ is invertible. Let $\hat{\boldsymbol{F}}_m(x)$ be defined on $\mathbb{R}$ as follows. For $x\in\mathbb{R}\setminus\left[-\frac{2m-1}{2}a,\frac{2m-1}{2}a\right]$, let $\hat{\boldsymbol{F}}_m(x)=0$ and for $x\in\left[-\frac{2m-1}{2}a,\frac{2m-1}{2}a\right]$ let $\hat{\boldsymbol{F}}_m(x)$ be defined by $\hat{\boldsymbol{F}}_m^t(x)=b\left(\hat{\boldsymbol{E}}_m^{-1}(x)\right)_m$, where $\hat{\boldsymbol{F}}_m(x)=\left[H_3(x+ka)\right]_{|k|\le m-1}$ and $\left(\hat{\boldsymbol{E}}_m^{-1}(x)\right)_m$ is the m^{th} column vector of the matrix $\hat{\boldsymbol{E}}_m^{-1}(x)$. Consequently, H_3 is a compactly supported and bounded function for which $\mathcal{G}(H_3,a,b)$ is a Bessel sequence. By construction, it also follows that B_3 and H_3 are dual windows.

3. Conclusion

We studied the fine structure of the Gabor frame generated by the B-spline B_3. It should be remembered that this structure determines all pairs $(a,b)\in\mathbb{R}_+^2$ for which the Gabor system $\mathcal{G}(B_3,a,b)$ forms a frame. We presented the known results of the Gabor frame set $\mathcal{F}(B_3)$ and we added a new set of points in the frame set of the 3-spline while building the compactly supported dual windows of B_3. All our results are obtained thanks to the partitioning of the domain in which the frame set is sought. This partitioning allowed us to establish a global approach, based on the study of the invertibility of a square matrix specific to each sub-domain, and which will be used to find other points belonging to the frame set of the B-spline B_3.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Christensen, O. (2003) An Introduction to Frames and Riesz Bases. Birkhäuser Boston Inc., Boston, MA. https://doi.org/10.1007/978-0-8176-8224-8

[2] Gröchenig, K. (2001) Foundations of Time-Frequency Analysis. Springer-Birkhäuser, New York. https://doi.org/10.1007/978-1-4612-0003-1

[3] Gröchenig, K. (2014) The Mystery of Gabor Frames. *Journal of Fourier Analysis and Applications*, **20**, 865-895. https://doi.org/10.1007/s00041-014-9336-3

[4] Kim, I. (2013) Gabor Frames with Trigonometric Spline Dual Windows. *Asian-European Journal of Mathematics*, **8**, Article ID: 1550072. https://doi.org/10.1142/S1793557115500722

[5] Kloos, T. and Stöckler, J. (2014) Zak Transforms and Gabor Frames of Totally Positive Functions and Exponential B-Splines. *Journal of Approximation Theory*, **184**, 209-237. https://doi.org/10.1016/j.jat.2014.05.010

[6] Laugesen, R.S. (2009) Gabor Dual Splines Windows. *Applied and Computational Harmonic Analysis*, **27**, 180-194. https://doi.org/10.1016/j.acha.2009.02.002

[7] Christensen, O., Kim, H.O. and Kim, R.Y. (2015) On Entire Functions Restricted to Intervals, Partition of Unities, and Dual Gabor Frames. *Applied and Computational Harmonic Analysis*, **38**, 72-86. https://doi.org/10.1016/j.acha.2014.03.005

[8] Gröchenig, K., Janssen, A.J.E.M., Kaiblinger, N. and Pfander, G. (2003) Note on B-Splines, Wavelet Scaling Functions, and Gabor Frames. *IEEE Transactions on Information Theory*, **49**, 3318-3320. https://doi.org/10.1109/TIT.2003.820022

[9] Christensen, O., Kim, H.O. and Kim, R.Y. (2010) Gabor Windows Supported on [-1] [1] and Compactly Supported Dual Windows. *Applied and Computational Harmonic Analysis*, **28**, 89-103.

[10] Christensen, O., Kim, H.O. and Kim, R.Y. (2012) Gabor Windows Supported on [-1, 1] and Dual Windows with Small Support. *Advances in Computational Mathematics*, **36**, 525-545. https://doi.org/10.1007/s10444-011-9189-0

[11] Christensen, O., Kim, H.O. and Kim, R.Y. (2015) On Gabor Frames Generated by Sign-Changing Windows and B-Splines. *Applied and Computational Harmonic Analysis*, **39**, 534-544. https://doi.org/10.1016/j.acha.2015.02.006

[12] Christensen, O. (2014) Six (Seven) Problems in Frame Theory. In: Zayed, A.I. and Schmeisser, G., Eds., *New Perspectives on Approximation and Sampling Theory*, *Applied and Numerical Harmonic Analysis*, Birkhäuser, Cham, 337-358. https://doi.org/10.1007/978-3-319-08801-3_14

[13] Dai, X. and Sun, Q. (2013) The *abc*-Problem for Gabor Systems. arXiv: 1304.7750v1 [cs.IT].

[14] Gröchenig, K. and Stöckler, J. (2013) Gabor Frames and Totally Positive Functions. *Duke Mathematical Journal*, **162**, 1003-1031. https://doi.org/10.1215/00127094-2141944

[15] Gu, Q. and Han, D. (2008) When a Characteristic Function Generates a Gabor Frame.

Applied and Computational Harmonic Analysis, **24**, 290-309.
https://doi.org/10.1016/j.acha.2007.06.005

[16] Janssen, A. and Strohmer, T. (2002) Hyperbolic Secants Yield Gabor Frames. *Applied and Computational Harmonic Analysis*, **12**, 259-267.
https://doi.org/10.1006/acha.2001.0376

[17] Janssen, A. (1996) Some Weyl-Heisenberg Frame Bound Calculations. *Indagationes Mathematicae*, **7**, 165-183. https://doi.org/10.1016/0019-3577(96)85088-9

[18] Janssen, A.J.E.M. (2003) Zak Transforms with Few Zeros and the Tie. In: Feichtinger, H.G. and Strohmer, T., Eds., *Advances in Gabor Analysis*, Birkhäuser, Boston, 31-70. https://doi.org/10.1007/978-1-4612-0133-5_3

[19] Janssen, A.J.E.M. (2003) On Generating Tight Gabor Frames at Critical Density. *Journal of Fourier Analysis and Applications*, **9**, 175-214.
https://doi.org/10.1007/s00041-003-0011-3

[20] Lyubarskii, Y. (1992) Frames in the Bargmann Space of Entire Functions. Entire and Subharmonic Functions. *Advances in Soviet Mathematics*, **11**, 167-180.
https://doi.org/10.1090/advsov/011/08

[21] Seip, K. (1992) Density Theorems for Sampling and Interpolation in the Bargmann-Fock Space I. *Journal für die reine und angewandte Mathematik*, **429**, 91-106.
https://doi.org/10.1515/crll.1992.429.91

[22] Seip, K., Wallstén, R. (1992) Density Theorems for Sampling and Interpolation in the Bargmann-Fock Space. II. *Journal für die reine und angewandte Mathematik*, **429**, 107-113. https://doi.org/10.1515/crll.1992.429.107

[23] Atindehou, A.G.D., Kouagou, Y.B. and Okoudjou, K.A. (2020) Frame Sets for a Class of Compactly Supported Continuous Functions. *Asian-European Journal of Mathematics*, **13**, Article ID: 2050093, 25 p. https://doi.org/10.1142/S179355712050093X

[24] Heil, C.E. and Walnut, D.F. (1989) Continuous and Discrete Wavelet Transforms. *SIAM Review*, **31**, 628-666. https://doi.org/10.1137/1031129

[25] Feitchinger, H.G. and Kaiblinger, N. (2004) Varying the Time-Frequency Lattice of Gabor Frames. *Transactions of the American Mathematical Society*, **356**, 2001-2023. https://doi.org/10.1090/S0002-9947-03-03377-4

[26] Ascensi, G., Feitchinger, H.G. and Kaiblinger, N. (2014) Dilation of the Weyl Symbol and the Balian-Low Theorem. *Transactions of the American Mathematical Society*, **366**, 3865-3880. https://doi.org/10.1090/S0002-9947-2013-06074-6

[27] Christensen, O., Kim, H.O. and Kim, R.Y. (2016) On Gabor Frame Set for Compactly Supported Continuous Functions. *Journal of Inequalities and Applications*, **2016**, Article No. 94. https://doi.org/10.1186/s13660-016-1021-4

[28] Lemvig, J. and Nielsen, H. (2016) Counterexamples to the B-Spline Conjecture for Gabor Frames. *Journal of Fourier Analysis and Applications*, **22**, 1440-1451.
https://doi.org/10.1007/s00041-016-9462-1

[29] Daubechies, I., Grossmann, A. and Meyer, Y. (1986) Painless Nonorthogonal Expansions. *Journal of Mathematical Physics*, **27**, 1271-1283.
https://doi.org/10.1063/1.527388

Appendix

Consider the first-order difference $\Delta_a B_3$ and the second-order difference $\Delta_a^2 B_3$ given respectively by

$\Delta_a B_3(x) = B_3(x) - B_3(x-a)$ and $\Delta_a^2 B_3(x) = B_3(x) - 2B_3(x-a) + B_3(x-2a)$

The following Lemma completes the Lemma 2.2 of [27] in the case of B_3.

Lemma 3. *Let* $0 < a < 3$. *Then* $\Delta_a^2 B_3(x) > 0$, *for all* $x \in \left(-\frac{3}{2}+a, \frac{3}{2}-\frac{2}{b}+a\right)$.

Proof. By definition of the functional space $V_a g$ and the Lemma 2.2 in [27], it is known that $\Delta_a^2 B_3(x) > 0$ for all $x \in \left[-\frac{3}{2}, -\frac{3}{4}+\frac{3a}{4}\right]$.

In particular, for $a \geq 1$ or $b \leq \frac{8}{9+a}$, $\left(-\frac{3}{2}+a, \frac{3}{2}-\frac{2}{b}+a\right) \subset \left[-\frac{3}{2}, -\frac{3}{4}+\frac{3a}{4}\right]$ while for $a < 1$ and $b > \frac{8}{9+a}$, $-\frac{3}{4}+\frac{3a}{4} < \frac{3}{2}-\frac{2}{b}+a$. In other words, to have that $\Delta_a^2 B_3(x) > 0$ on $\left(-\frac{3}{2}+\frac{3a}{4}, \frac{3}{2}-\frac{2}{b}+a\right)$, it suffices to show that, for $a < 1$ and $b > \frac{8}{9+a}$,

$$\Delta_a^2 B_3(x) > 0, \forall x \in \left(-\frac{3}{4}+\frac{3a}{4}, \frac{3}{2}-\frac{2}{b}+a\right).$$

Otherwise, we only show that for $\frac{3}{4} \leq a < 1$ and $\frac{8}{9+a} < b \leq \frac{4}{3+2a}$;

$$\Delta_a^2 B_3(x) > 0, \forall x \in \left(-\frac{3}{4}+\frac{3a}{4}, \frac{3}{2}-\frac{2}{b}+a\right) \subset \left(-\frac{3}{16}, 0\right].$$

Let $f(x) := \Delta_a^2 B_3(x)$. It is easy to see that for all $x \in \left(-\frac{3}{4}+\frac{3a}{4}, \frac{3}{2}-\frac{2}{b}+a\right)$,

$$\begin{aligned} f(x) &= B_3(x) - 2B_3(x-a) \\ &= -x^2 + \frac{3}{4} - (x-a)^2 - 3(x-a) - \frac{9}{4} \\ &= -2x^2 + (2a-3)x - a^2 + 3a - \frac{3}{2}. \end{aligned}$$

One has $f'\left(-\frac{3}{16}\right) = 2a - \frac{9}{4} < 0$ et $f(0) = -a^2 + 3a - \frac{3}{2} > 0$. Thus $f(x) > 0$, $\forall x \in \left(-\frac{3}{16}, 0\right]$. Consequently,

$$\forall x \in \left(-\frac{3}{4}+\frac{3a}{4}, \frac{3}{2}-\frac{2}{b}+a\right), \Delta_a^2 B_3(x) > 0$$

as wanted.

The following Lemma shows the invertibility of the matrix $\boldsymbol{A}_{44}^2(x)$.

Lemma 4. *Let* $a \in \left[\frac{3}{4}, \frac{3}{2}\right]$, $b \in \left[\frac{3}{3+a}, \frac{4}{3+2a}\right]$ *and* $x \in \left(-\frac{3}{2}+\frac{1}{b}; \frac{3}{2}-\frac{2}{b}+a\right)$.

Then $\left|\boldsymbol{A}_{44}^{2}(x)\right|>0$.

Proof. Let $\boldsymbol{L}_2(x):=\left|\boldsymbol{A}_{44}^{2}(x)\right|$ and $\boldsymbol{L}_2^{i,j}(x)$ denote the ij^{th} minor of $\boldsymbol{L}_2(x)$, the determinant of the matrix obtained by removing the i^{th} row and the j^{th} column from $\boldsymbol{L}_2(x)$. We have respectively

(1) $B_3(x)>B_3(x-a)$ and $B_3(x)>B_3(x+a)$;

(2) $\boldsymbol{L}_2^{3,2}(x)>0$ and $\boldsymbol{L}_2^{3,4}(x)>0$;

(3) $\boldsymbol{L}_2^{33}(x)>\boldsymbol{L}_2^{3,2}(x)+\boldsymbol{L}_2^{3,4}(x)$.

Combining (1), (2) and (3), we have for all $x\in\left(-\frac{3}{2}+\frac{1}{b};\frac{3}{2}-\frac{2}{b}+a\right)$,

$$\begin{aligned}\boldsymbol{L}_2(x)&=-B_3(x-a)\cdot\boldsymbol{L}_2^{3,2}(x)+B_3(x)\cdot\boldsymbol{L}_2^{33}(x)-B_3(x+a)\cdot\boldsymbol{L}_2^{3,4}(x)\\&>\left[B_3(x)-B_3(x-a)\right]\cdot\boldsymbol{L}_2^{3,2}(x)+\left[B_3(x)-B_3(x+a)\right]\cdot\boldsymbol{L}_2^{3,4}(x)>0.\end{aligned}$$

This implies that for all $x\in\left(-\frac{3}{2}+\frac{1}{b};\frac{3}{2}-\frac{2}{b}+a\right)$, $\left|\boldsymbol{A}_{44}^{2}(x)\right|>0$.

We have for all $x\in\left(-\frac{3}{2}+\frac{1}{b};\frac{3}{2}-\frac{2}{b}+a\right)$, $x-a<x<0<x+a$ and $-x-a<x$. Thus $B_3(x-a)<B_3(x)$ and $B_3(x)>B_3(x+a)$. On the other hand,

$$\boldsymbol{L}_2^{3,2}(x)=B_3\left(x+\frac{2}{b}-2a\right)B_3\left(x+\frac{1}{b}\right)B_3\left(x-\frac{1}{b}+a\right)>0$$

and $$\boldsymbol{L}_2^{3,4}(x)=B_3\left(x-\frac{1}{b}\right)\left|\boldsymbol{A}_{22}^{2}(x)\right|>0.$$

Then (1) and (2) hold.

One has

$$\begin{aligned}\boldsymbol{L}_2^{33}(x)-\boldsymbol{L}_2^{3,2}(x)-\boldsymbol{L}_2^{3,4}(x)=&\left[B_3\left(x-\frac{1}{b}+a\right)-B_3\left(x-\frac{1}{b}\right)\right]\left|\boldsymbol{A}_{22}^{2}(x)\right|\\&-B_3\left(x+\frac{2}{b}-2a\right)B_3\left(x+\frac{1}{b}\right)B_3\left(x-\frac{1}{b}+a\right)\end{aligned}$$

where

$$B_3\left(x-\frac{1}{b}\right)=\frac{1}{2}\left(x-\frac{1}{b}\right)^2+\frac{3}{2}\left(x-\frac{1}{b}\right)+\frac{9}{8},\quad B_3\left(x+\frac{1}{b}\right)=\frac{1}{2}\left(x-\frac{1}{b}\right)^2-\frac{3}{2}\left(x+\frac{1}{b}\right)+\frac{9}{8}$$

and

$$B_3\left(x-\frac{1}{b}+a\right)=\begin{cases}\frac{1}{2}\left(x-\frac{1}{b}+a\right)^2+\frac{3}{2}\left(x-\frac{1}{b}+a\right)+\frac{9}{8} & \text{if } a\in\left[\frac{3}{4},1\right]\\-\left(x-\frac{1}{b}+a\right)^2+\frac{3}{4} & \text{if } a\in\left[1;\frac{3}{2}\right].\end{cases}$$

Let $g(x):=\left[B_3\left(x-\frac{1}{b}+a\right)-B_3\left(x-\frac{1}{b}\right)\right]-B_3\left(x+\frac{1}{b}\right)$. It is easy to remark that the function g is strictly increasing on $\left(-\frac{3}{2}+\frac{1}{b};\frac{3}{2}-\frac{2}{b}+a\right)$ and

$g\left(-\frac{3}{2}+\frac{1}{b}\right)>0$ where

$$g\left(-\frac{3}{2}+\frac{1}{b}\right)=\begin{cases}\frac{1}{2}\left(-3+a+\frac{2}{b}\right)\left(3+a-\frac{2}{b}\right) & \text{if } a\in\left[\frac{3}{4},1\right]\\ -a^2+3a-\frac{2}{b^2}+\frac{6}{b}-6 & \text{if } a\in\left[1;\frac{3}{2}\right].\end{cases}$$

Consequently $B_3\left(x-\frac{1}{b}+a\right)-B_3\left(x-\frac{1}{b}\right)>B_3\left(x+\frac{1}{b}\right)$.

Let $h(x):=\left|A_{22}^2(x)\right|-B_3\left(x+\frac{2}{b}-2a\right)B_3\left(x-\frac{1}{b}+a\right)$ where

$$A_{22}^2(x)=\begin{pmatrix}B_3\left(x+\frac{2}{b}-2a\right) & B_3\left(x+\frac{2}{b}-a\right)\\ B_3\left(x+\frac{1}{b}-2a\right) & B_3\left(x+\frac{1}{b}-a\right)\end{pmatrix}.$$

We obtain

$$\min_x h(x)=h\left(\frac{3}{2}-\frac{2}{b}+a\right)=B_3\left(\frac{3}{2}-a\right)\left[B_3\left(\frac{3}{2}-\frac{1}{b}\right)-B_3\left(\frac{3}{2}-\frac{3}{b}+2a\right)\right]>0$$

because $\frac{3}{2}-\frac{3}{b}+2a<-\frac{3}{2}+\frac{1}{b}\Rightarrow B_3\left(\frac{3}{2}-\frac{3}{b}+2a\right)<B_3\left(\frac{3}{2}-\frac{1}{b}\right)$. Consequently (3) holds.

Therefore $\left|A_{22}^2(x)\right|>B_3\left(x+\frac{2}{b}-2a\right)B_3\left(x-\frac{1}{b}+a\right)$. Hence $L_2^{33}(x)>L_2^{3,2}(x)+L_2^{3,4}(x)$.

Proof of Lemma 2. Throughout this proof, we use the Lemma 1 and Remark 1 without notify it.

(a) Let $k\in\{1,\cdots,m-1\}$. In the first time, we consider the matrix $A_{22}^k(x)$ given by

$$A_{22}^k(x)=\begin{pmatrix}B_3\left(x+\frac{k}{b}-ka\right) & B_3\left(x+\frac{k}{b}-(k-1)a\right)\\ B_3\left(x+\frac{k-1}{b}-ka\right) & B_3\left(x+\frac{k-1}{b}-(k-1)a\right)\end{pmatrix}.$$

We prove that for all $k\in\{1,\cdots,m-1\}$, $B_3\left(x+\frac{k}{b}-ka\right)>B_3\left(x+\frac{k}{b}-(k-1)a\right)$. For this, we know that $-\frac{3}{2}<x+\frac{k}{b}-ka<\frac{3}{2}$ and $x+\frac{k}{b}-(k-1)a>0$, therefore we have two different cases.

* If $x+\frac{k}{b}-ka>0$, we have $x+\frac{k}{b}-ka<x+\frac{k}{b}-(k-1)a$ and using the strict decreasing of B_3 on $\left[0,\frac{3}{2}\right]$, one has $B_3\left(x+\frac{k}{b}-ka\right)>B_3\left(x+\frac{k}{b}-(k-1)a\right)$.

* If $x+\frac{k}{b}-ka\le 0$, then $-x-\frac{k}{b}+ka\ge 0$ and

$$-x-\frac{k}{b}+ka-\left(x+\frac{k}{b}-(k-1)a\right)=-2x-\frac{2k}{b}+(2k-1)a\le 2k\left(a-\frac{1}{b}\right)<0.$$

Thus $-x-\frac{k}{b}+ka<x+\frac{k}{b}-(k-1)a$ and using the fact that B_3 is symmetric around the origin and strict decreasing on $\left[0,\frac{3}{2}\right]$, one has

$B_3\left(x+\frac{k}{b}-ka\right)>B_3\left(x+\frac{k}{b}-(k-1)a\right)$.

Next, we prove that for all $k\in\{1,\cdots,m-2\}$,

$B_3\left(x+\frac{k-1}{b}-ka\right)<B_3\left(x+\frac{k-1}{b}-(k-1)a\right)$. We also know that $-\frac{3}{2}<x+\frac{k-1}{b}-(k-1)a<\frac{3}{2}$ and $x+\frac{k-1}{b}-ka<0$, therefore we can consider the following two cases.

* If $x+\frac{k-1}{b}-(k-1)a\le 0$, on has $x+\frac{k-1}{b}-ka<x+\frac{k-1}{b}-(k-1)a$. Therefore by the strict increasing of B_3 on $\left[-\frac{3}{2};0\right]$, we obtain

$B_3\left(x+\frac{k-1}{b}-ka\right)<B_3\left(x+\frac{k-1}{b}-(k-1)a\right)$.

* If $x+\frac{k-1}{b}-(k-1)a>0$, then $-x-\frac{k-1}{b}+(k-1)a\le 0$ and $-x-\frac{k-1}{b}+(k-1)a-\left(x+\frac{k-1}{b}-ka\right)=-2x-\frac{2(k-1)}{b}+(2k-1)a\ge 0$. Indeed,

$$\begin{aligned}
&-2x-\frac{2(k-1)}{b}+(2k-1)a\ge-\frac{2(k-1)}{b}+(2k-1)a\\
&\ge-(k-1)\frac{3+(2m-3)a}{m-1}+(2k-1)a=-\frac{3(k-1)}{m-1}+\frac{k+m-2}{m-1}a\\
&\ge-\frac{3(k-1)}{m-1}+\frac{k+m-2}{m-1}\times\frac{3(m-3)}{2(m-2)}=\frac{3(m-2-k)}{2(m-2)}\ge 0.
\end{aligned}$$

Thus $x+\frac{k-1}{b}-ka\le-x-\frac{k-1}{b}+(k-1)a$ and using the fact that B_3 is symmetric around the origin and strict increasing on $\left[-\frac{3}{2};0\right]$, one has

$B_3\left(x+\frac{k-1}{b}-ka\right)\le B_3\left(x+\frac{k-1}{b}-(k-1)a\right)$.

All in all, we conclude that $\left|A_{22}^{k}(x)\right|>0$ for all $k\in\{1,\cdots,m-2\}$.

For $k=m-1$, we consider the matrix $A_{22}^{m-1}(x)$ given by

$$A_{22}^{m-1}(x)=\begin{pmatrix} B_3\left(x+\frac{m-1}{b}-(m-1)a\right) & B_3\left(x+\frac{m-1}{b}-(m-2)a\right)\\ B_3\left(x+\frac{m-2}{b}-(m-1)a\right) & B_3\left(x+\frac{m-2}{b}-(m-2)a\right)\end{pmatrix}.$$

We know respectively that $B_3\left(x+\frac{m-1}{b}-(m-1)a\right)>0$,

$B_3\left(x+\frac{m-2}{b}-(m-1)a\right)>0$, $B_3\left(x+\frac{m-2}{b}-(m-2)a\right)>0$,

$$B_3\left(x+\frac{m-1}{b}-(m-1)a\right) > B_3\left(x+\frac{m-1}{b}-(m-2)a\right) \text{ and}$$

$$B_3\left(x+\frac{m-1}{b}-(m-2)a\right) = \begin{cases} B_3\left(x+\frac{m-1}{b}-(m-2)a\right) > 0 & \text{if } x \in \left[-\frac{a}{2};\frac{3}{2}-\frac{m-1}{b}+(m-2)a\right) \\ 0 & \text{if } x \in \left[\frac{3}{2}-\frac{m-1}{b}+(m-2)a;0\right] \end{cases}$$

It is easy to see that if $x \in \left[\frac{3}{2}-\frac{m-1}{b}+(m-2)a;0\right]$, then $\left|A_{22}^{m-1}(x)\right| > 0$. To end, we prove that for all $x \in \left[-\frac{a}{2};\frac{3}{2}-\frac{m-1}{b}+(m-2)a\right)$, $\left|A_{22}^{m-1}(x)\right| > 0$.

We also know that $-\frac{3}{2} < x+\frac{m-2}{b}-(m-2)a < \frac{3}{2}$ and $x+\frac{m-2}{b}-(m-1)a < 0$, therefore we can consider the following two cases.

*If $x+\frac{m-2}{b}-(m-2)a \le 0$, one has $x+\frac{m-2}{b}-(m-1)a < x+\frac{m-2}{b}-(m-2)a$. Therefore by the strict increasing of B_3, we obtain $B_3\left(x+\frac{m-2}{b}-(m-1)a\right) < B_3\left(x+\frac{m-2}{b}-(m-2)a\right)$.

*If $x+\frac{m-2}{b}-(m-2)a > 0$, then $-x-\frac{m-2}{b}+(m-2)a \le 0$ and

$$-x-\frac{m-2}{b}+(m-2)a-\left(x+\frac{m-2}{b}-(m-1)a\right) = -2x-\frac{2(m-2)}{b}+(2m-3)a > -3+\frac{2}{b}+a \ge 0.$$ Thus

$x+\frac{m-2}{b}-(m-1)a < -x-\frac{m-2}{b}+(m-2)a$ and using the fact that B_3 is symmetric around the origin and strict increasing, one has

$$B_3\left(x+\frac{m-2}{b}-(m-1)a\right) \le B_3\left(x+\frac{m-2}{b}-(m-2)a\right).$$

In the second time, we consider, for all $k \in \{1,\cdots,m-1\}$, the matrix $A_{33}^{k}(x)$ given by

$$A_{33}^{k}(x) = \begin{pmatrix} B_3\left(x+\frac{k}{b}-ka\right) & B_3\left(x+\frac{k}{b}-(k-1)a\right) & 0 \\ B_3\left(x+\frac{k-1}{b}-ka\right) & B_3\left(x+\frac{k-1}{b}-(k-1)a\right) & B_3\left(x+\frac{k-1}{b}-(k-2)a\right) \\ 0 & B_3\left(x+\frac{k-2}{b}-(k-1)a\right) & B_3\left(x+\frac{k-2}{b}-(k-2)a\right) \end{pmatrix}$$

For $x \in \left[-\frac{a}{2};-\frac{3}{2}-\frac{k-2}{b}+(k-1)a\right] \cup \left[\frac{3}{2}-\frac{k}{b}+(k-1)a;0\right]$, we have

$B_3\left(x+\frac{k-2}{b}-(k-1)a\right)=0$ or $B_3\left(x+\frac{k}{b}-(k-1)a\right)=0$ and hence

$$\left|A_{33}^{k}(x)\right|=B_3\left(x+\frac{k-2}{b}-(k-2)a\right)\left|A_{22}^{k}(x)\right|>0$$
$$\text{or } \left|A_{33}^{k}(x)\right|=B_3\left(x+\frac{k}{b}-ka\right)\left|A_{22}^{k-1}(x)\right|>0.$$

Let $x\in\left(-\frac{3}{2}-\frac{k-2}{b}+(k-1)a;\frac{3}{2}-\frac{k}{b}+(k-1)a\right)$.

We proved previously that $B_3\left(x+\frac{k}{b}-ka\right)>B_3\left(x+\frac{k}{b}-(k-1)a\right)$ for all $k\in\{1,\cdots,m-1\}$.

Nest, we prove that $B_3\left(x+\frac{k-1}{b}-ka\right)<B_3\left(x+\frac{k-1}{b}-(k-1)a\right)$ and $B_3\left(x+\frac{k-1}{b}-(k-1)a\right)>B_3\left(x+\frac{k-1}{b}-(k-2)a\right)$.

We know for all $k\in\{1,\cdots,m-1\}$, $-\frac{3}{2}<x+\frac{k-1}{b}-(k-1)a<\frac{3}{2}$, $x+\frac{k-1}{b}-ka<0$ and $x+\frac{k-1}{b}-(k-2)a>0$, therefore we have different following cases.

*If $x+\frac{k-1}{b}-(k-1)a\le 0$, on has $x+\frac{k-1}{b}-ka<x+\frac{k-1}{b}-(k-1)a$. Therefore by the strict increasing of B_3 on $\left[-\frac{3}{2};0\right]$, we obtain $B_3\left(x+\frac{k-1}{b}-ka\right)<B_3\left(x+\frac{k-1}{b}-(k-1)a\right)$.

*If $x+\frac{k-1}{b}-(k-1)a>0$, then $-x-\frac{k-1}{b}+(k-1)a<0$ and $-x-\frac{k-1}{b}+(k-1)a-\left(x+\frac{k-1}{b}-ka\right)=-2x-\frac{2(k-1)}{b}+(2k-1)a>0$. Indeed,

$$x\in\left(-\frac{3}{2}-\frac{k-2}{b}+(k-1)a;\frac{3}{2}-\frac{k}{b}+(k-1)a\right)$$
$$\Rightarrow -2x-\frac{2(k-1)}{b}+(2k-1)a>-3+\frac{2}{b}+a>0.$$

Therefore $x+\frac{k-1}{b}-ka<-x-\frac{k-1}{b}+(k-1)a$ and consequently we have

$$B_3\left(x+\frac{k-1}{b}-ka\right)<B_3\left(x+\frac{k-1}{b}-(k-1)a\right).$$

*If $x+\frac{k-1}{b}-(k-1)a>0$, we have $x+\frac{k-1}{b}-(k-1)a<x+\frac{k-1}{b}-(k-2)a$ and then $B_3\left(x+\frac{k-1}{b}-(k-1)a\right)>B_3\left(x+\frac{k-1}{b}-(k-2)a\right)$.

*If $x+\frac{k-1}{b}-(k-1)a\le 0$, then $-x-\frac{k-1}{b}+(k-1)a\ge 0$ and

$-x-\frac{k-1}{b}+(k-1)a-\left(x+\frac{k-1}{b}-(k-2)a\right)=-2x-\frac{2(k-1)}{b}+(2k-3)a<0$ because

$$x\in\left(-\frac{3}{2}-\frac{k-2}{b}+(k-1)a;\frac{3}{2}-\frac{k}{b}+(k-1)a\right)\Rightarrow -2x-\frac{2(k-1)}{b}+(2k-3)a<3-\frac{2}{b}-a<0.$$

Thus $-x-\frac{k-1}{b}+(k-1)a<x+\frac{k-1}{b}-(k-2)a$ and using the fact that B_3 is symmetric around the origin and strict decreasing on $\left[0,\frac{3}{2}\right]$, one has

$$B_3\left(x+\frac{k-1}{b}-(k-1)a\right)>B_3\left(x+\frac{k-1}{b}-(k-2)a\right).$$

Let us prove that $B_3\left(x+\frac{k-2}{b}-(k-1)a\right)<B_3\left(x+\frac{k-2}{b}-(k-2)a\right)$ for all $k\in\{1,\cdots,m-1\}$.

One has $B_3\left(x-\frac{1}{b}\right)<B_3\left(x-\frac{1}{b}+a\right)$ because $-\frac{3}{2}<x-\frac{1}{b}<x-\frac{1}{b}+a<0$.

Let $k\in\{2,\cdots,m-1\}$. We known that $-\frac{3}{2}<x+\frac{k-2}{b}-(k-2)a<\frac{3}{2}$ and $x+\frac{k-2}{b}-(k-1)a<0$, therefore we have two cases.

*If $x+\frac{k-2}{b}-(k-2)a\leq 0$, then $x+\frac{k-2}{b}-(k-1)a<x+\frac{k-2}{b}-(k-2)a$ and therefore $B_3\left(x+\frac{k-2}{b}-(k-1)a\right)<B_3\left(x+\frac{k-2}{b}-(k-2)a\right)$.

*If $x+\frac{k-2}{b}-(k-2)a>0$, then $-x-\frac{k-2}{b}+(k-2)a<0$ and

$-x-\frac{k-2}{b}+(k-2)a-\left(x+\frac{k-2}{b}-ka\right)$

$=-2x-\frac{2(k-2)}{b}-(2k-3)a>-3+\frac{4}{b}-a>0$. Therefore

$x+\frac{k-2}{b}-ka<-x-\frac{k-2}{b}+(k-2)a$ and then

$$B_3\left(x+\frac{k-2}{b}-(k-1)a\right)<B_3\left(x+\frac{k-2}{b}-(k-2)a\right).$$

Let

$$p(x)=\begin{vmatrix} B_3\left(x+\frac{k}{b}-ka\right) & 0 \\ 0 & B_3\left(x+\frac{k-2}{b}-(k-2)a\right) \end{vmatrix} - \begin{vmatrix} B_3\left(x+\frac{k}{b}-(k-1)a\right) & 0 \\ B_3\left(x+\frac{k-2}{b}-(k-1)a\right) & B_3\left(x+\frac{k-2}{b}-(k-2)a\right) \end{vmatrix}$$

$$-\begin{vmatrix} B_3\left(x+\frac{k}{b}-ka\right) & B_3\left(x+\frac{k}{b}-(k-1)a\right) \\ 0 & B_3\left(x+\frac{k-2}{b}-(k-1)a\right) \end{vmatrix}.$$

A direct computation shows that

$$p(x)=B_3\left(x+\frac{k}{b}-ka\right)B_3\left(x+\frac{k-2}{b}-(k-2)a\right)-B_3\left(x+\frac{k}{b}-(k-1)a\right)$$
$$\times B_3\left(x+\frac{k-2}{b}-(k-2)a\right)-B_3\left(x+\frac{k}{b}-ka\right)B_3\left(x+\frac{k-2}{b}-(k-1)a\right)$$
$$=-\Delta_a B_3\left(x+\frac{k}{b}-(k-1)a\right)\Delta_a B_3\left(x+\frac{k-2}{b}-(k-2)a\right)$$
$$+\Delta_a B_3\left(x+\frac{k}{b}-(k-2)a\right)\Delta_a B_3\left(x+\frac{k-2}{b}-(k-1)a\right)$$
$$=\Delta_a B_3\left(-x-\frac{k}{b}+ka\right)\Delta_a B_3\left(x+\frac{k-2}{b}-(k-2)a\right)-\Delta_a B_3\left(-x-\frac{k}{b}+(k-1)a\right)$$
$$\times\Delta_a B_3\left(x+\frac{k-2}{b}-(k-1)a\right)\ \left(\text{one uses } \Delta_a B_3(x)=-\Delta_a B_3(-x+a)\right)$$

Hence to have $p(x)>0$, it suffices to show that for all

$$x\in\left(-\frac{3}{2}-\frac{k-2}{b}+(k-1)a;\frac{3}{2}-\frac{k}{b}+(k-1)a\right),$$

$$\begin{cases}\Delta_a B_3\left(x+\frac{k-2}{b}-(k-2)a\right)>\Delta_a B_3\left(x+\frac{k-2}{b}-(k-1)a\right)\\ \Delta_a B_3\left(-x-\frac{k}{b}+ka\right)>\Delta_a B_3\left(-x-\frac{k}{b}+(k-1)a\right)\end{cases}$$
$$\Leftrightarrow\begin{cases}\Delta_a^2 B_3\left(x+\frac{k-2}{b}-(k-2)a\right)>0\\ \Delta_a^2 B_3\left(-x-\frac{k}{b}+ka\right)>0\end{cases}$$

This means precisely that $\Delta_a^2 B_3(x)>0$, $x\in\left(-\frac{3}{2}+a,\frac{3}{2}-\frac{2}{b}+a\right)$ which is obtained in Lemma 3.

b) Let $k\in\{2,\cdots,m-1\}$ and consider the matrix

$$A_{44}^k(x)=\begin{pmatrix} B_3\left(x+\frac{k}{b}-ka\right) & B_3\left(x+\frac{k}{b}-(k-1)a\right) & 0 & 0\\ B_3\left(x+\frac{k-1}{b}-ka\right) & B_3\left(x+\frac{k-1}{b}-(k-1)a\right) & B_3\left(x+\frac{k-1}{b}-(k-2)a\right) & 0\\ 0 & B_3\left(x+\frac{k-2}{b}-(k-1)a\right) & B_3\left(x+\frac{k-2}{b}-(k-2)a\right) & B_3\left(x+\frac{k-2}{b}-(k-3)a\right)\\ 0 & 0 & B_3\left(x+\frac{k-3}{b}-(k-2)a\right) & B_3\left(x+\frac{k-3}{b}-(k-3)a\right)\end{pmatrix}.$$

*If $3-\frac{3}{b}+a\le 0$, then $\frac{3}{2}-\frac{k}{b}+(k-1)a\le-\frac{3}{2}-\frac{k-3}{b}+(k-2)a$ and therefore

$\forall\, x \in \left[-\frac{a}{2};0\right]$, $B_3\left(x+\frac{k-3}{b}-(k-2)a\right)=0$ or $B_3\left(x+\frac{k}{b}-(k-1)a\right)=0$. Thus

$$\left|\boldsymbol{A}_{44}^{k}(x)\right| = B_3\left(x+\frac{k-3}{b}-(k-3)a\right)\cdot\left|\boldsymbol{A}_{33}^{k}(x)\right|>0$$

$$\text{or } \left|\boldsymbol{A}_{44}^{k}(x)\right| = B_3\left(x+\frac{k}{b}-ka\right)\cdot\left|\boldsymbol{A}_{33}^{k-1}(x)\right|>0.$$

*If $3-\frac{3}{b}+a>0$, then $-\frac{3}{2}-\frac{k-3}{b}+(k-2)a<\frac{3}{2}-\frac{k}{b}+(k-1)a$.

When $x\in\left[-\frac{a}{2};-\frac{3}{2}-\frac{k-3}{b}+(k-2)a\right]\cup\left[\frac{3}{2}-\frac{k}{b}+(k-1)a;0\right]$, then $B_3\left(x+\frac{k-3}{b}-(k-2)a\right)=0$ or $B_3\left(x+\frac{k}{b}-(k-1)a\right)=0$ and therefore

$$\left|\boldsymbol{A}_{44}^{k}(x)\right| = B_3\left(x+\frac{k-3}{b}-(k-3)a\right)\cdot\left|\boldsymbol{A}_{33}^{k}(x)\right|>0$$

$$\text{or } \left|\boldsymbol{A}_{44}^{k}(x)\right| = B_3\left(x+\frac{k}{b}-ka\right)\cdot\left|\boldsymbol{A}_{33}^{k-1}(x)\right|>0.$$

We finish by proving that for all $x\in\left(-\frac{3}{2}-\frac{k-3}{b}+(k-2)a;\frac{3}{2}-\frac{k}{b}+(k-1)a\right)$, $\left|\boldsymbol{A}_{44}^{k}(x)\right|>0$. We observe that for $k=3$ and $x\in\left(-1+a;1-\frac{3}{b}+2a\right)$, then $\left|\boldsymbol{A}_{44}^{3}(x)\right|=\left|\boldsymbol{A}_{44}^{2}\left(x+\frac{1}{b}-a\right)\right|$ and for $k\in\{4,\cdots,m-1\}$ and $x\in\left(-\frac{3}{2}-\frac{k-3}{b}+(k-2)a;\frac{3}{2}-\frac{k}{b}+(k-1)a\right)$, then $\left|\boldsymbol{A}_{44}^{k}(x)\right|=\left|\boldsymbol{A}_{44}^{3}\left(x+\frac{k-3}{b}-(k-3)a\right)\right|>0$. So we only prove that for all $x\in\left(-\frac{3}{2}+\frac{1}{b};\frac{3}{2}-\frac{2}{b}+a\right)$, $\left|\boldsymbol{A}_{44}^{2}(x)\right|>0$. Using the condition $3-\frac{3}{b}+a>0$, we consider $a\in\left[\frac{3}{4},\frac{3}{2}\right]$ and $b\in\left[\frac{3}{3+a},\frac{4}{3+2a}\right]$. In other words, by the Lemma 4, we have this result.