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On the Uniqueness of the Limiting Solution to a Strongly Coupled Singularly Perturbed Elliptic System

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Abstract

This article is concerned with a strongly coupled elliptic system modeling the steady state of two or more populations that compete in some regions. We prove the uniqueness of the limiting configuration as the competing rate tends to infinity, under suitable conditions. The proof relies on properties of limiting solution and Maximum principle.

Keywords

Uniqueness, Spatial Segregation, Strongly Coupled Elliptic System, Free Boundary Problems

1. Introduction

In this paper, we consider the following strongly coupled system of elliptic equations:

$$\begin{cases} -\Delta\left[\left(d_i+\sum_j \beta_{ij}u_j^k\right)u_i^k\right]=\left(a_i-b_iu_i^k\right)u_i^k-ku_i^k\sum_{j\neq i}u_j^k & \text{in } \Omega, \\ u_i^k=\phi_i, \quad i=1,\cdots,m & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where u_i denotes the density of the i-th population, $i=1,\cdots,m$, $m\geq 2$ is the number of the species and Ω is a bounded domain in $\mathbb{R}^n\,(n\geq 1)$ with smooth boundary. d_i is the diffusion rate, a_i the intrinsic growth rate, b_i the intraspecific competition rate and b_{ij} the interspecific competition rate, β_{ii} represents the self-diffusion rate, and $\beta_{ij}\,(i\neq j)$ represents the cross-diffusion rate, ϕ_i are given Lipschitz continuous functions on $\partial\Omega$, which satisfy $\phi_i\geq 0$ and $\phi_i\phi_j=0$ for $i\neq j$. k is a free positive parameter, which is sufficiently large (or its limit at $k=\infty$).

System (1.1) represents a model of the steady state of m competing species with self- and cross-population pressures. In the case when $\beta_{ij} \equiv 0$ for every i and j, system (1.1) is the classic Lotka-Volterra competition model:

$$\begin{cases} -d_i \Delta u_i^k = \left(a_i - b_i u_i^k\right) u_i^k - k u_i^k \sum_{j \neq i} u_j^k & \text{in } \Omega, \\ u_i^k = \phi_i, \quad i = 1, \cdots, m & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

While if $\beta_{ij} > 0$ for some i, j, the system becomes strongly coupled. System (1.1) (or its parabolic case) has been investigated by many workers [1]-[6], and various existing results have been developed. In particular, when $m = 2$, Lou and Ni [2] characterized the existence of nonconstant positive solutions both for the small and large competition cases, while those in [4] [5] were concerned with the existence of positive solutions in relation to a pair of curves in the (a_1, a_2)-plane for both large and small cross-diffusion cases. For the existing results concerning the case when $m \geq 3$, we refer to [6] and references therein.

According to Gause's principle of competitive exclusion, two competing species cannot coexist under strong competition. The migration or the spatial distribution changes the situation and all the species survive but have disjoint habits, which is called spatial segregation [7]. To investigate such a phenomenon, we will focus on the so called strong competition regime, that is when the parameter k diverges to $+\infty$, while the positive coefficients b_{ij} remain fixed.

In the classic Lotka-Volterra competition model (1.2), it is proved that k-dependent solutions $u_k = \left(u_1^k, \cdots, u_m^k\right)$ of system (1.2) satisfy uniform bounds in Hölder norms and converge, up to a subsequence, to some limit $u = (u_1, \cdots, u_m)$, having disjoint supports: $u_i u_j = 0$ for $i \neq j$ [8]. In the limiting configuration, the common zero set $\Gamma(u) = \{u = 0\}$ can be considered as a free boundary (see for example [8]-[13]). When $b_{ij} = b_{ji}$ for all i and j (symmetric interactions case), it is proved that the free boundary consists of two parts: a regular set, which is a $C^{1,\alpha}$ locally smooth hypersurface, and a singular set of Hausdorff dimension not greater than $n-2$; furthermore, in dimension 2, then free boundary consists in a locally finite collection of curves meeting with equal angles at a locally finite number of singular points, see for example [8] [9] [14]. Unlike the symmetric case, the asymmetric case (*i.e.* when $b_{ij} \neq b_{ji}$ for some i, j) shows the emergence of spiraling nodal curves, still meeting at locally isolated points with finite vanishing order [15].

A further related problem is the study of the uniqueness and least energy property of the limiting configuration as $k \to +\infty$. In the case of three species and in dimension 2, Conti *et al.* [16] proved the uniqueness and least energy properties for the limiting state. That is, the solution of system (1.2) (when $a_i = b_i = 0$) converges, as $k \to +\infty$, to the minimizer of a variational problem. In [13], Wang and Zhang generalized the result to arbitrary dimensions and arbitrary number of species. In [17], Arakelyan and Bozorgnia also proved the uniqueness of the limiting solution to system (2).

On the other hand, coming back to the strongly coupled system, Zhou *et al.* [18] [19] study the asymptotic behavior of solutions to system (1.1). They obtained the similar spatial segregation results and established the uniform C^α ($0<\alpha<1$) bounds for solutions to system (1.1).

In this paper, we continue the study of system (1.1), we are concerned with the uniqueness of the limiting configuration of system (1.1). In order to simplify the notations, throughout the paper we assume $b_{ij}=b_{ji}\equiv 1$, for $i\neq j$. We only consider nonnegative solutions, that is, those $u_i^k\geq 0$ in its domain for all *i*. First we observe that, as proved in [19], the segregated limit $u=(u_1,\cdots,u_m)$ satisfies in distributional sense that

$$\begin{cases} -\Delta\left[\left(d_i+\beta_{ii}u_i\right)u_i\right]\leq\left(a_i-b_iu_i\right)u_i & \text{in } \Omega,\\ -\Delta\left[\left(d_i+\beta_{ii}u_i\right)u_i-\sum_{j\neq i}\left(d_j+\beta_{jj}u_j\right)u_j\right] & \\ \geq\left(a_i-b_iu_i\right)u_i-\sum_{j\neq i}\left(a_j-b_ju_j\right)u_j & \text{in } \Omega,\\ -\Delta\left[\left(d_i+\beta_{ii}u_i\right)u_i\right]=\left(a_i-b_iu_i\right)u_i & \text{in } \{u_i>0\},\\ u_i=\phi_i & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Define the singular space

$$\mathcal{U}:=\left\{\left(u_1,\cdots,u_m\right)\in\left(H^1(\Omega)\right)^m: u_i\geq 0, u_i\big|_{\partial\Omega}=\phi_i \text{ and } u_iu_j=0 \text{ for } i\neq j\right\}.$$

Our result is as follows.

Theorem 1.1. *Assume that*

$$\max_i\left\{a_i/d_i\right\}<\lambda_1(\Omega), \tag{1.4}$$

where $\lambda_1(\Omega)$ *denotes the first eigenvalue of the operator* $-\Delta$ *with zero Dirchlet boundary condition on* Ω. *Then there exists a unique vector* $(u_1,\cdots,u_m)\in\mathcal{U}$ *satisfying* (1.3)

We note that Theorem 1.1 has already been proved in [19], where the uniqueness, also the least energy properties for the limiting state has been established. Their method originally stated in [13], is based on computing the derivative of the energy functional with respect to the geodesic homotopy between u and a comparison to an energy minimizing map v with same boundary values. Our proof is different from the one in [13] [19]. In fact, our method follows the mainstream of [17], based on the properties of limiting solutions and Maximum principle. Compared with the work of [19], we in fact give a new proof of the uniqueness of the limiting configuration. Our proof doesn't require regular results of the free boundary. So in this sense, our proof is straightforward and simple.

Note that the study of strong-competition limits in corresponding elliptic or parabolic system is of interest not only for questions of spatial segregation in population, as here and in [20] [21], but also is key to the understanding of phase separation of Gross Pitaevskii systems of modeling Bose-Einstein condensates, see [22]-[27] and reference therein. Furthermore, the study on other aspects of segregation triggered by strong competition, starting from two pioneering papers by Dancer and Du in [20] [21], is now very vast; besides the papers quoted above, we mention [28] [29] [30] [31] for analogue studies in nonlocal contexts, [32] [33] for long-range interaction models.

The rest of the paper is organized as follows: In section 2, we introduce a transformation and recall some preliminary results, which are essential to the proof of the main results. In Section 3, we prove the uniqueness of the system (1.1) in the limiting case as k tends to infinity.

2. Some Preliminary Results

In this section, we mention some known results for the solutions of system (1.1), which play an important role in our study. To begin with, for every index i, we define

$$z_i^k = \left(d_i + \sum_{j=1}^{m} u_j^k\right) u_i^k. \tag{2.1}$$

Then the Jacobian determinant

$$J = \frac{\partial\left(z_1^k,\cdots,z_m^k\right)}{\partial\left(u_1^k,\cdots,u_m^k\right)} = \begin{vmatrix} d_1 + 2\beta_{11}u_1^k + \sum_{j\neq 1}\beta_{1j}u_j^k & \beta_{12}u_1^k & \cdots & \beta_{1m}u_1^k \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m1}u_m^k & \beta_{m2}u_m^k & \cdots & d_m + 2\beta_{mm}u_m^k + \sum_{j\neq m}\beta_{mj}u_j^k \end{vmatrix} > d_1 d_2 \cdots d_m > 0.$$

So there exist inverse functions $u_i^k = f_i\left(z_1^k,\cdots,z_m^k\right)$ for $i = 1,\cdots,m$, which are continuous and have continuous partial derivatives.

To simplify the notations we denote by $f_i\left(z_k\right) = f_i\left(z_1^k,\cdots,z_M^k\right)$ and using (2.1) we may write system (1.1) in the following equivalent form:

$$\begin{cases} -\Delta z_i^k = \left(a_i - b_i f_i\left(z_k\right)\right) f_i\left(z_k\right) - k f_i\left(z_k\right)\sum_{j\neq i} f_j\left(z_k\right) & \text{in } \Omega \\ v_i^k = \left(d_i + \beta_{ii}\phi_i\right)\phi_i & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Now we recall some estimates and compactness properties of solutions to system (1.1).

Lemma 2.1 ([19]) *Let* $\boldsymbol{u}_k = \left(u_1^k,\cdots,u_M^k\right)$ *be a nonnegative solution of* (1.1) *for some* $k \in \mathbb{N}$, *and* $\boldsymbol{z}_k = \left(z_1^k,\cdots,z_M^k\right)$ *be defined as in* (2.1). *Then* $\boldsymbol{z}_k$ *is a nonnegative solution of* (2.2), *and for every* $0 < \alpha < 1$, *there exists a constant* $C_\alpha > 0$ *independent of k such that*

$$\left\|\boldsymbol{u}_k\right\|_{C^{0,\alpha}(\bar{\Omega})} \le C_\alpha, \quad \left\|\boldsymbol{z}_k\right\|_{C^{0,\alpha}(\bar{\Omega})} \le C_\alpha.$$

Moreover, there exists $u = \left(u_1,\cdots,u_m\right) \in \left(H^1(\Omega)\right)^m$ such that for all $i = 1, 2,\cdots,m$,

1) up to subsequences, $u_i^k \to u_i$ in $H^1(\Omega)\cap C^{0,\alpha}(\Omega)$;

2) if we define for each index i:

$$z_i = \left(d_i + \beta_{ii}u_i\right)u_i, \tag{2.3}$$

then up to subsequences, $z_i^k \to z_i$ in $H^1(\Omega) \cap C^{0,\alpha}(\Omega)$;

3) $u_i u_j = 0$ and $z_i z_j = 0$ in Ω, for $i \neq j$. Furthermore, in distributional sense, z_i satisfies

$$\begin{cases} -\Delta z_i \leq h_i(z_i) & \text{in } \Omega, \\ -\Delta\left(z_i - \sum_{j\neq i} z_j\right) \geq h_i(z_i) - \sum_{j\neq i} h_j(z_j) & \text{in } \Omega, \\ -\Delta z_i = h_i(z_i) & \text{in } \{z_i > 0\}, \\ z_i = (d_i + \beta_{ii}\phi_i)\phi_i & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where

$$h_i(s) = \frac{\sqrt{4\beta_{ii}s + d_i^2} - d_i}{2\beta_{ii}} \left(a_i - b_i \frac{\sqrt{4\beta_{ii}s + d_i^2} - d_i}{2\beta_{ii}} \right). \tag{2.5}$$

Remark 2.1. *By* (2.4) *and Theorem* 8.2 *in* [14], *we have that each element of* $z = (z_1, \cdots, z_m)$ *is actually global Lipschitz continuous on* Ω.

3. Uniqueness of the Limiting Configuration

In this section, we prove Theorem 1.1. We perform a change of variable in order to deal with the problem in a different setting. Let $u = (u_1, \cdots, u_m)$ and $z = (z_1, \cdots, z_m)$ be as the statement in Section 2. Assume that (1.4) holds. We define

$$\lambda := \max_i \left\{ \sup_{0 < s \leq \|z\|_{L^\infty(\Omega)}} \frac{|h_i(s)|}{s} \right\} \tag{3.1}$$

with $h_i(s)$ be given in (2.5). It is obvious that for each *i*, h_i is Lipschitz continuous and $h_i(0) = 0$, so (3.1) is well defined. By assumption (1.4), we have $\lambda \leq \max_i \left\{ \frac{a_i}{d_i} \right\} < \lambda_1(\Omega)$, and, this implies the existence of a positive function $p(x) \in C^2(\Omega)$ such that

$$\begin{cases} -\Delta p = \lambda p & \text{in } \Omega, \\ p > 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Indeed, the monotonicity of the first eigenvalue of the Dirichlet problem with respect to the domain implies that there exists $\Omega_1 \supsetneqq \Omega$ such that $\lambda = \lambda_1(\Omega_1) < \lambda_1(\Omega)$. Let $\eta \in H_0^1(\Omega_1)$ be the corresponding eigenfunction of the operator $-\Delta$ with zero Dirchlet boundary condition on Ω_1. Then $\eta > 0$ in Ω_1, and by the elliptic regularity theory $\eta \in C^2(\Omega_1)$. So if we let $p(x)$ be the restriction of $\eta(x)$ to Ω, then $p(x) \in C^2(\bar{\Omega})$ (note that $\partial\Omega$ is regular) and satisfies (3.2). In particular, there exists a constant $p_0 > 0$ such that $p(x) > p_0$ for all $x \in \Omega$. We now define

$$v_i = u_i(d_i + \beta_{ii}u_i)/p = z_i(x)/p,\ i = 1, \cdots, m, \tag{3.3}$$

then $v_i = 0$ if and only if $z_i = 0$. By Remark 2.1, for every index *i*, v_i is Lipschitz continuous and, by Lemma 2.1, v_i satisfies in distributional sense that

$$
\begin{cases}
-\text{div}\left(p^2\nabla v_i\right) \le ph_i(pv_i) - \lambda p^2 v_i & \text{in } \Omega, \\
-\text{div}\left(p^2\nabla\left(v_i - \sum_{j\neq i} v_j\right)\right) & \\
\ge p\left[h_i\left(pv_i\right) - \sum_{j\neq i} h_j\left(pv_j\right)\right] - \lambda p^2\left(v_i - \sum_{j\neq i} v_j\right) & \text{in } \Omega, \\
-\text{div}\left(p^2\nabla v_i\right) = ph_i\left(pv_i\right) - \lambda p^2 v_i & \text{in } \{v_i > 0\}, \\
v_i = \left(d_i + \beta_{ii}\phi_i\right)\phi_i / p & \text{on } \partial\Omega.
\end{cases} \tag{3.4}
$$

By the definition of $v = \left(v_1, \cdots, v_m\right)$, we have $v_i v_j \equiv 0$ for $i \neq j$. In this setting, we consider the corresponding singular space

$$
\mathcal{S} := \Big\{\left(v_1, \cdots, v_m\right) \in \left(H^1(\Omega)\right)^m : v_i \ge 0, v_i\big|_{\partial\Omega} = \left(d_i + \beta_{ii}\phi_i\right)\phi_i / p \text{ and } v_i v_j = 0 \text{ for } i \neq j\Big\}.
$$

By above construction, we know that if there exists a unique vector $\left(v_1, \cdots, v_m\right) \in \mathcal{S}$ satisfying (3.4), the uniqueness for the original system (1.3) then follows by the definition of the change of the variables, and the proof of Theorem 1.1 is complete. In the following, we focus on the analysis of system (3.4). To begin with, for every index *i*, we denote

$$
\hat{w}_i(x) := w_i(x) - \sum_{p\neq i} w_p(x). \tag{3.5}
$$

Lemma 3.1. *Let two elements* $\left(v_1, \cdots, v_m\right)$ *and* $\left(w_1, \cdots, w_m\right)$ *belong to* $\mathcal{S}$ *and satisfying* (3.4). *Then the following equation for each* $1 \le i \le m$ *holds*:

$$
\max_{\Omega}\left(\hat{v}_i(x) - \hat{w}_i(x)\right) = \max_{\{v_i(x) \le w_i(x)\}}\left(\hat{v}_i(x) - \hat{w}_i(x)\right).
$$

Proof We argue by contradiction. Let there exists some i_0 such that

$$
\max_{\Omega}\left(\hat{v}_{i_0} - \hat{w}_{i_0}\right) = \max_{\{v_{i_0} > w_{i_0}\}}\left(\hat{v}_{i_0} - \hat{w}_{i_0}\right) > \max_{\{v_{i_0} \le w_{i_0}\}}\left(\hat{v}_{i_0} - \hat{w}_{i_0}\right) \tag{3.6}
$$

Assume $\mathcal{D} = \left\{x \in \Omega : v_{i_0}(x) > w_{i_0}(x)\right\}$, then in $\mathcal{D}$ we have

$$
\begin{cases}
-\text{div}\left(p^2\nabla\hat{v}_{i_0}\right) = ph_i\left(pv_{i_0}\right) - \lambda p^2 v_{i_0}, \\
-\text{div}\left(p^2\nabla\hat{w}_{i_0}\right) \ge p\left[h_i\left(pw_{i_0}\right) - \sum_{j\neq i_0} h_j\left(pw_j\right)\right] - \lambda p^2\left(w_{i_0} - \sum_{j\neq i_0} w_j\right).
\end{cases} \tag{3.7}
$$

We claim that:

$$
-\text{div}\left[p^2\nabla\left(\hat{v}_{i_0} - \hat{w}_{i_0}\right)\right] \le 0.
$$

In fact, by (3.7)

$$
\begin{aligned}
&-\text{div}\left[p^2\nabla\left(\hat{v}_{i_0} - \hat{w}_{i_0}\right)\right] \\
&\le ph_i\left(pv_{i_0}\right) - \lambda p^2 v_{i_0} - p\left[h_i\left(pw_{i_0}\right) - \sum_{j\neq i_0} h_j\left(pw_j\right)\right] + \lambda p^2\left(w_{i_0} - \sum_{j\neq i_0} w_j\right) \\
&= \left[ph_i\left(pv_{i_0}\right) - ph_i\left(pw_{i_0}\right)\right] + \left[p\sum_{j\neq i_0} h_j\left(pw_j\right) - \lambda p^2 \sum_{j\neq i_0} w_j\right] - \lambda p^2\left(v_{i_0} - w_{i_0}\right) \\
&\doteq I_1 + I_2 - I_3.
\end{aligned} \tag{3.8}
$$

Since h_i is Lipschitz continuous and $h_i(0)=0$, by the definition of λ (see (3.1)) we have

$$I_1 = ph_i\left(pv_{i_0}\right) - ph_i\left(pw_{i_0}\right) \le \lambda p\left(pv_{i_0} - pw_{i_0}\right) = I_3,$$

Similarly

$$\begin{aligned} I_2 &= p\sum_{j\ne i_0} h_j\left(pw_j\right) - \lambda p^2 \sum_{j\ne i_0} w_j \\ &= p\sum_{j\ne i_0}\left[h_j\left(pw_j\right) - \lambda pw_j\right] \\ &\le p\sum_{j\ne i_0}\left(\lambda pw_j - \lambda pw_j\right) = 0, \end{aligned}$$

and the claim follows. We can now use the weak maximum principle to conclude that

$$\max_D\left(\hat{v}_{i_0} - \hat{w}_{i_0}\right) \le \max_{\partial D}\left(\hat{v}_{i_0} - \hat{w}_{i_0}\right) \le \max_{\{v_{i_0}=w_{i_0}\}}\left(\hat{v}_{i_0} - \hat{w}_{i_0}\right) \le \max_{\{v_{i_0}\le w_{i_0}\}}\left(\hat{v}_{i_0} - \hat{w}_{i_0}\right),$$

which contradicts (3.6). Then we can interchange the role of $\hat{v}_i$ and $\hat{w}_i$. Thus, we also have

$$\max_{\bar{\Omega}}\left(\hat{w}_i(x) - \hat{v}_i(x)\right) = \max_{\{w_i(x)\le v_i(x)\}}\left(\hat{w}_i(x) - \hat{v}_i(x)\right),$$

for all $1\le i\le m$, and we complete the proof of Lemma 3.1. □

In view of Lemma 3.1 we define the following quantities

$$P := \max_{1\le i\le m}\left(\max_{\bar{\Omega}}\left(\hat{v}_i(x) - \hat{w}_i(x)\right)\right) = \max_{1\le i\le m}\left(\max_{\{v_i\le w_i\}}\left(\hat{v}_i(x) - \hat{w}_i(x)\right)\right),$$

$$Q := \max_{1\le i\le m}\left(\max_{\bar{\Omega}}\left(\hat{w}_i(x) - \hat{v}_i(x)\right)\right) = \max_{1\le i\le m}\left(\max_{\{w_i\le v_i\}}\left(\hat{w}_i(x) - \hat{v}_i(x)\right)\right).$$

Lemma 3.2. *Let two elements* $(v_1,\cdots,v_m)$ *and* $(w_1,\cdots,w_m)$ *belong to* $\mathcal{S}$ *and satisfying* (3.4). *We set P and Q as defined above. If* $P>0$ *is attained for some index* $1\le i\le m$, *then we have* $P=Q>0$. *Moveover, there exist another index* $j_0\ne i_0$ *and a point* $x_0\in\Omega$, *such that*:

$$P = Q = \max_{\{v_{i_0}\le w_{i_0}\}}\left(\hat{v}_{i_0} - \hat{w}_{i_0}\right) = \max_{\{v_{i_0}=w_{i_0}=0\}}\left(\hat{v}_{i_0} - \hat{w}_{i_0}\right) = w_{j_0}(x_0) - w_{j_0}(x_0).$$

Proof Let the maximum $P>0$ be attained for the i_0^{th} component. According to the previous lemma, we know that $\left(\hat{v}_{i_0}(x) - \hat{w}_{i_0}(x)\right)$ attains its maximum on the set $\left\{v_{i_0}(x)\le w_{i_0}(x)\right\}$. Let that maximum point be $x^*\in\left\{v_{i_0}(x)\le w_{i_0}(x)\right\}$. So, if $\hat{v}_{i_0}(x^*) - \hat{w}_{i_0}(x^*) = P > 0$, then we have

$$v_{i_0}(x^*) = w_{i_0}(x^*) = 0.$$

Indeed, if $v_{i_0}(x^*) = w_{i_0}(x^*) > 0$, then in the light of disjointness property of the components of v_i and w_i we get $P = \hat{v}_{i_0}(x^*) - \hat{w}_{i_0}(x^*) = v_{i_0}(x^*) - w_{i_0}(x^*) = 0$ which is a contradiction. If $v_{i_0}(x^*) < w_{i_0}(x^*)$, then again due to the disjointness of the densities v_i, w_i, we have

$$0 < P = \hat{v}_{i_0}(x^*) - \hat{w}_{i_0}(x^*) = \hat{v}_{i_0}(x^*) - w_{i_0}(x^*) \le v_{i_0}(x^*) - w_{i_0}(x^*) < 0.$$

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This again leads to a contradiction. Therefore $v_{i_0}(x^*) = w_{i_0}(x^*) = 0$.

Now assume by contradiction that $Q \le 0$. Then by definition of Q we should have

$$\hat{w}_j(x) \le \hat{v}_j(x), \quad \forall x \in \Omega, j = 1, \cdots, m.$$

This apparently yields

$$w_j(x) \le v_j(x), \quad \forall x \in \Omega, j = 1, \cdots, m.$$

If $w_j(x) > v_j(x)$, then $w_j(x) = \hat{w}_j(x) \le \hat{v}_j(x) = v_j(x) - \sum_{h \ne j} v_h \le v_j$, obtaining a contradiction.

Let $\mathcal{D}_{i_0} = \{v_{i_0}(x) = w_{i_0}(x) = 0\}$, then we have

$$0 < P = \max_{\mathcal{D}_{i_0}} \left(\hat{v}_{i_0}(x) - \hat{w}_{i_0}(x)\right) = \max_{\mathcal{D}_{i_0}} \left(\sum_{j \ne i_0} \left(w_j(x) - v_j(x)\right) \right) \le 0.$$

This contradiction implies that $Q > 0$. By analogous proof, one can see that if P be non-positive then Q will be non-positive as well. Next, assume the maximum P is attained at a point $x_0 \in \mathcal{D}_{i_0}$. Then we get

$$\begin{aligned} 0 < P &= \hat{v}_{i_0}(x_0) - \hat{w}_{i_0}(x_0) = \left(v_{i_0}(x_0) - w_{i_0}(x_0)\right) + \sum_{j \ne i_0} \left(w_j(x_0) - v_j(x_0)\right) \\ &= \sum_{j \ne i_0} \left(w_j(x_0) - v_j(x_0)\right). \end{aligned}$$

This shows that

$$\sum_{j \ne i_0} w_j(x_0) = \sum_{j \ne i_0} v_j(x_0) + P > 0.$$

Since $(w_1, \cdots, w_m) \in \mathcal{S}$, then there exists $j_0 \ne i_0$ such that $w_{j_0}(x_0) > 0$. This implies

$$\begin{aligned} 0 < P &= \hat{v}_{i_0}(x_0) - \hat{w}_{i_0}(x_0) = w_{j_0}(x_0) - \sum_{j \ne i_0} v_j(x_0) \\ &= \hat{w}_{j_0}(x_0) - \sum_{j \ne i_0} v_j(x_0) + 2v_{j_0}(x_0) - 2v_{j_0}(x_0) \\ &= \hat{w}_{j_0}(x_0) - \sum_{j \ne i_0, j_0} v_j(x_0) + v_{j_0}(x_0) - 2v_{j_0}(x_0) \\ &= \hat{w}_{j_0}(x_0) - \hat{v}_{j_0}(x_0) - 2v_{j_0}(x_0) \\ &\le \hat{w}_{j_0}(x_0) - \hat{v}_{j_0}(x_0) \le Q. \end{aligned}$$

The same argument shows that $Q \le P$ which yields $P = Q$. Hence, we can write

$$P = w_{j_0}(x_0) - \sum_{j \ne i_0} v_j(x_0) = \hat{w}_{j_0}(x_0) - \hat{v}_{j_0}(x_0) = Q.$$

This gives us $2\sum_{j \ne j_0} v_j(x_0) = 0$, and therefore

$$v_j(x_0) = 0, \ \forall j \ne j_0,$$

which completes the last statement of the proof. □

We are ready to the proof of Theorem 1.1. As already mentioned, it is sufficient to prove the following unique result for system (4).

Theorem 3.1. *There exists a unique vector* $(v_1, \cdots, v_m) \in \mathcal{S}$, *which satisfies* *system* (3.4).

Proof Let $u=(u_1,\cdots,u_m)$ and $u'=(u'_1,\cdots,u'_m)$ be two m-tuples of the limiting solutions of system (1.1) as $k\to+\infty$. Then we define

$$v_i=u_i\left(d_i+\beta_{ii}u_i\right)/p \text{ and } w_i=u'_i\left(d_i+\beta_{ii}u'_i\right)/p, i=1,\cdots,m.$$

It is now clear that $v=(v_1,\cdots,v_m)$ and $w=(w_1,\cdots,w_m)$ are belong to the class $\mathcal{S}$ and satisfy (3.4). For them, we set P and Q as above. Then, we consider two cases $P\le 0$ and $P>0$. If we assume that $P\le 0$ then Lemma 3.2 implies that $Q\le 0$. This leads to

$$0\le -Q\le \hat{v}_i(x)-\hat{w}_i(x)\le P\le 0,$$

for every $1\le i\le m$, and $x\in\Omega$. This provides that

$$\hat{v}_i(x)-\hat{w}_i(x),\ i=1,\cdots,m,$$

which in turn implies that

$$v_i(x)=w_i(x).$$

Now, suppose $P>0$, we show that this case leads to a contradiction. Let the value P is attained for some i_0, then due to Lemma 3.2 there exist $x_0\in\Omega$ and $j_0\ne i_0$ such that:

$$0<P=Q=\hat{v}_{i_0}(x_0)-\hat{w}_{i_0}(x_0)=\max_{\{v_{i_0}=w_{i_0}=0\}}\left(\hat{v}_{i_0}(x)-\hat{w}_{i_0}(x)\right)=w_{j_0}(x_0)-v_{j_0}(x_0).$$

Let Γ be a fixed curve starting at x_0 and ending on the boundary of Ω. Since Ω is connected, then one can always choose such a curve belonging to Ω. By the disjointness and smoothness of v_{j_0} and u_{j_0}, there exists a ball centered at x_0, and with radius r_0 (r_0 depends on x_0) which we denote it $B_{r_0}(x_0)$, such that

$$w_{j_0}(x)-v_{j_0}(x)>0 \text{ in } B_{r_0}(x_0).$$

This yields

$$-\operatorname{div}\left(p^2\nabla\left(\hat{w}_{j_0}(x)-\hat{v}_{j_0}(x)\right)\right)\le 0 \text{ in } B_{r_0}(x_0).$$

The maximum principle implies that

$$\max_{B_{r_0}(x_0)}\left(\hat{w}_{j_0}(x)-\hat{v}_{j_0}(x)\right)=\max_{\partial B_{r_0}(x_0)}\left(\hat{w}_{j_0}(x)-\hat{v}_{j_0}(x)\right)\le P.$$

On the other hand, in view of Lemma 3.2 we have

$$\hat{w}_{j_0}(x)-\hat{v}_{j_0}(x)=w_{j_0}(x_0)-v_{j_0}(x_0)=P,$$

which implies that P is attained at the interior point $x_0\in B_{r_0}(x_0)$. Thus,

$$\hat{w}_{j_0}(x)-\hat{v}_{j_0}(x)\equiv P>0 \text{ in } \overline{B_{r_0}(x_0)}.$$

Next let $x_1\in\Gamma\cap\partial B_{r_0}(x_0)$. We get $\hat{w}_{j_0}(x_1)-\hat{v}_{j_0}(x_1)=P>0$, which leads to $w_{j_0}(x_1)\ge v_{j_0}(x_1)$. We proceed as follows: If $w_{j_0}(x_1)>v_{j_0}(x_1)$, then as above

$$w_{j_0}(x)>v_{j_0}(x) \text{ in } B_{r_1}(x_1).$$

This in turn implies

$$-\operatorname{div}\left(p^2\nabla\left(\hat{w}_{j_0}(x)-\hat{v}_{j_0}(x)\right)\right)\le 0 \text{ in } B_{r_1}(x_1).$$

Again following the maximum principle and recalling that $\hat{w}_{j_0}(x_1)-\hat{v}_{j_0}(x_1)=P$ we conclude that

$$\hat{w}_{j_0}(x)-\hat{v}_{j_0}(x)=P>0 \text{ in } \overline{B_{r_1}(x_1)}.$$

If $w_{j_0}(x_1)=v_{j_0}(x_1)$, then clearly the only possibility is $w_{j_0}(x_1)=v_{j_0}(x_1)=0$. Thus

$$0<P=\hat{w}_{j_0}(x_1)-\hat{v}_{j_0}(x_1)=\left(w_{j_0}(x_1)-v_{j_0}(x_1)\right)+\sum_{j\ne j_0}\left(v_j(x_1)-w_j(x_1)\right)$$
$$=\sum_{j\ne j_0}\left(v_j(x_1)-w_j(x_1)\right).$$

Following the lines of the proof of Lemma 3.2, we find some $k_0\ne j_0$, such that

$$P=v_{k_0}(x_1)-w_{k_0}(x_1)=\hat{v}_{k_0}(x_1)-\hat{w}_{k_0}(x_1).$$

It is easy to see that there exists a ball $B_{r_1}(x_1)$ (without loss of generality one keeps the same notation)

$$-\operatorname{div}\left(p^2\nabla\left(\hat{v}_{k_0}(x)-\hat{w}_{k_0}(x)\right)\right)\le 0 \text{ in } B_{r_1}(x_1).$$

In view of the maximum principle and above steps we obtain:

$$\hat{v}_{k_0}(x)-\hat{w}_{k_0}(x)=P>0 \text{ in } \overline{B_{r_1}(x_1)}.$$

Then we take $x_2\in\Gamma\cap\partial B_{r_1}(x_1)$ such that x_1 stands between the points x_0 and x_2 along the given curve Γ. According to the previous arguments for the point x_2 we will find an index $l_0\in\{1,\cdots,m\}$ and corresponding ball $\overline{B_{r_2}(x_2)}$, such that

$$\left|\hat{v}_{l_0}(x)-\hat{w}_{l_0}(x)\right|=P \text{ in } \overline{B_{r_2}(x_2)}.$$

We continue this way and obtain a sequence of points x_n along the curve Γ, which are getting closer to the boundary of Ω. Since for all $j=1,\cdots,m$ and $x\in\partial\Omega$ we have

$$\hat{v}_j(x)-\hat{w}_j(x)=\hat{w}_j(x)-\hat{v}_j(x)=0,$$

then obviously after finite steps N we find the point x_N, which will be very close to the $\partial\Omega$ and for all $j=1,\cdots,m$

$$\left|\hat{v}_j(x_N)-\hat{w}_j(x_N)\right|<P/2.$$

On the other hand, according to our construction for the point x_N, there exists an index $1\le j_N\le m$ such that

$$\left|\hat{v}_{j_N}(x_N)-\hat{w}_{j_N}(x_N)\right|=P,$$

which is a contradiction. This completes the proof of the uniqueness. □

4. Conclusions and Further Works

The study of the asymptotic behavior of singular perturbed equations and systems of elliptic or parabolic type is very broad and subject of research. In this paper, we study a strongly coupled elliptic system arising in competing models in population dynamics. We give an alternative proof of the uniqueness of the limiting configuration as $k \to +\infty$ under suitable conditions. We remark that the approach here is different from the one in [19]. Our proof doesn't require regular results of the free boundary. So in this sense, our proof is straightforward and simple.

Finally, we mention that there are many interesting problems for further study. Note that we prove the uniqueness of the limiting solutions to a strongly coupled elliptic system, naturally to ask whether this result can be extended to the corresponding parabolic system? Up to our knowledge, the uniform Hölder bounds for parabolic setting is unknown, and both the asymptotics and the qualitative properties of the limit segregated profiles remain a challenge, this will be the object of a forthcoming paper.

Founding

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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