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On Some New Properties for the Gamma Functions

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this paper will be presented some new properties in form of the inequalities related to gamma function, product of two gamma functions and some of them will be presented in the form of infinite series and limit where are included the Euler number and the Euler-Mascheroni constant.

Keywords: Gamma function; series; Euler number; Euler-Mascheroni constant.

Mathematics Subject Classification 2020: 65D20.

1 Introduction

Special functions represent a special class of mathematical functions, which take place in different branches of mathematics, such as mathematical analysis, functional analysis, differential equations, complex analysis, they also find application outside mathematics, as well as in physics and various branches in engineering.

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There is no definite strict definition of what special functions actually represent, but there are a large number of functions that are accepted as such.

Some special functions are taken as solutions of differential equations, some others are defined through parametric integrals, and some as functional series.

The gamma function was first defined by Euler, who on January 8, 1730 wrote a letter to Christian Goldbach, where he defined the gamma function as follows:

$$\Gamma(x) = \int_0^1 (-logt)^{x-1} dt$$

where $x > 0$.

The notation $\Gamma(x)$ and the name gamma function was introduced by Legendre(1752-1833) ([1]).

The beta function was also given by Euler, who is also known as the integral of the first type, while the gamma integral of the second type.

In this paper we give and verify some properties related to the gamma function, where some of them include the Euler's number and the Euler-Mascheroni constant.

2 Auxiliary Facts

In this unit will be given some known results-lemmas that will be used in proofs of our main results.

Isaac Barrow (1670) and Jacob Bernoulli (1689) proved the inequality, which now has the name Bernoulli inequality ([2]):

Lemma 1 (Bernoulli's inequality) For $r \geq 0$ and $x \geq -1$, follows the inequality $(1+x)^r \geq 1 + rx$

Lema2 ([3]) $\Gamma(z+1) = z\Gamma(z), Re(z) > 0$

In [4] are proven the following relations:

Lemma 3 ([4])The following relation holds:

$$\sum_{n=1}^{\infty} \frac{q^n}{(2!)^{\frac{q-q^3}{1-q^n}} \cdot (3!)^{\frac{q^2-q^3}{1-q^n}} \cdot \ldots \cdot (n!)^{\frac{q^{n-1}-q^n}{1-q^n}}} < e^{1+q}, where \quad 0 < q < 1.$$

Lemma4 ([4]) The following relation holds:

$$e = \sqrt{\sum_{n=0}^{\infty} \frac{(2n+3)2^{2n}}{(2n+1)!}}$$

Lemma5 ([5]) For $0 \leq a \leq 1$, the following relation holds:

$$2^{a-1} \leq \Gamma(1+a) \leq 1$$

Lemma6 ([6])The function

$$F(x) = \frac{ln\Gamma(1+x)}{xlnx}$$

is rigorously increasing in the interval $(1, \infty)$.

Lemma 7([7]) For every gamma function the following relation is valid:

$$\Gamma(x) = \frac{1}{x} - \gamma + O(x), \text{ when } x \to 0.$$

3 Main Results

Proposition 1
For $p > 0$, holds the inequality: $\Gamma(p+1) > \frac{1}{p+1}$

Proof

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx, p > 0$$

$$\Gamma(2+p) = \int_0^\infty x^{1+p} e^{-x} dx = \int_0^\infty (1 + x - 1)^{1+p} e^{-x} dx \geq (\; Lema\ 1)$$

$$\geq \int_0^\infty \big(1 + (x-1)(p+1)\big) e^{-x} dx = \int_0^\infty e^{-x} dx + \int_0^\infty (x-1)(p+1) e^{-x} dx =$$

$$= \Gamma(1) + (p+1)\left(\int_0^\infty x e^{-x} dx - \int_0^\infty e^{-x} dx\right) =$$

$$= \Gamma(1) + (p+1)\big(\Gamma(2) - \Gamma(1)\big) = 1$$

Finally,

$$\Gamma(2+p) > 1$$
$$\Gamma(1+p+1) > 1$$
$$(p+1)\Gamma(p+1) > 1$$
$$\Gamma(p+1) > \frac{1}{p+1}$$

Proposition 2 The following relation holds:

$$\sum_{n=1}^{\infty} \frac{\Gamma^n(1+a)}{(2!)^{\frac{\Gamma(1+a)-\Gamma^2(1+a)}{1-\Gamma^n(1+a)}} \cdot (3!)^{\frac{\Gamma^2(1+a)-\Gamma^3(1+a)}{1-\Gamma^n(1+a)}} \cdot \ldots \cdot (n!)^{\frac{\Gamma^{n-1}(1+a)-\Gamma^n(1+a)}{1-\Gamma^n(1+a)}}}$$

$$< e^{\Gamma(1+a)} \sqrt{\sum_{n=0}^{\infty} \frac{(2n+3)2^{2n}}{(2n+1)!}} \;, where\;\; 0 < a < 1$$

Proof. Now, for $0 \leq a \leq 1 \Longrightarrow 0 \leq \Gamma(1+a) \leq 1$, according to lemma 5, we denote

$q = \Gamma(1+a)$, and using and substituting this in the lemma 3, then instead of the number "e" in the right hand of the inequality we replace the expression in lemma 4, is obtained the stated result in proposition 2.

Proposition 3 For $a > b > e$, $b^a > a^b$.

Proof. For this known result, we give an alternate proof. Let us take

$$f(x) = \frac{x}{lnx}$$

Which after taking the first derivative we get

$$f'(x) = \frac{lnx - 1}{(lnx)^2}$$

so, for $x > e$, the function is incresing, respectively, for

$$a > b > e \Rightarrow \qquad f(a) > f(b) \Rightarrow \frac{a}{lna} > \frac{b}{lnb} \Rightarrow b^a > a^b.$$

Proposition 4 For $e < b < a$, the relation holds: $log_{\Gamma(a^b)}\Gamma(b^a) > log_a$b.

Proof. Using the proposition 3 and lemma 6 we get:

$$\frac{ln(\Gamma(1 + b^a))}{b^a lnb^a} > \frac{ln(\Gamma(1 + a^b))}{a^b lna^b}$$

$$\frac{1}{b^a} + \frac{1}{ab^a}\frac{ln(\Gamma(b^a))}{lnb} > \frac{1}{a^b} + \frac{1}{ba^b}\frac{ln(\Gamma(a^b))}{lna}$$

$$\frac{1}{ab^a}\frac{ln\big(\Gamma(b^a)\big)}{lnb} - \frac{1}{ba^b}\frac{ln(\Gamma(a^b))}{lna} > \frac{1}{a^b} - \frac{1}{b^a} > 0\,,$$

$since\ b^a > a^b$

$$\frac{1}{ab^a}\frac{ln(\Gamma(b^a))}{lnb} > \frac{1}{ba^b}\frac{ln(\Gamma(a^b))}{lna}$$

$$\frac{ln(\Gamma(b^a))}{lnb} > \frac{ab^a}{ba^b}\frac{ln(\Gamma(a^b))}{lna} > \frac{ln(\Gamma(a^b))}{lna}$$

respectively

$$log_{\Gamma(a^b)}\Gamma(b^a) > log_a \text{b}.$$

Proposition 5 Let $n \in \mathbb{N}_0$ and $z \notin \mathbb{Z}$ than holds the relation

$$\frac{\Gamma(2n + 2 - z)}{\Gamma(2n - z)} = (2n + 1 - z)(2n - z)$$

Proof Using the relations

$$\Gamma(2n + 2 - z)\Gamma(2n - z) =$$
$$= (2n + 1 - z)\Gamma(2n + 1 - z)\Gamma(2n - z) =$$
$$= (2n + 1 - z)(2n - z)\Gamma^2(2n - z)$$

Thus,

$$\frac{\Gamma(2n + 2 - z)}{\Gamma(2n - z)} = (2n + 1 - z)(2n - z)$$

Substituting $n = 0$ and $z = -\frac{1}{2}$ we get :

$$\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{3}{2}\cdot\frac{1}{2} = \frac{3}{4} \Rightarrow \Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}$$

Similarly, for $n = 1$ and $z = -\frac{3}{2}$ we get

$$\frac{\Gamma\left(4+\frac{3}{2}\right)}{\Gamma\left(2+\frac{3}{2}\right)} = \left(3+\frac{3}{2}\right)\left(2+\frac{3}{2}\right) \Rightarrow \frac{\Gamma\left(\frac{11}{2}\right)}{\Gamma\left(\frac{7}{2}\right)} = \frac{63}{4}$$

$$\Rightarrow \Gamma\left(\frac{11}{2}\right) = \frac{63}{4}\cdot\frac{5}{2}\cdot\Gamma\left(\frac{5}{2}\right) = \frac{945}{32}\sqrt{\pi}.$$

Proposition6 Let $\lim_{n\to\infty} a_n = \infty$ such that $a_n \geq 1, \forall n \in \mathbb{N}$, then $\lim_{n\to\infty} \Gamma^{a_n}\left(1+\frac{1}{a_n}\right) = e^{-\gamma}$, where e represents Euler number, and γ represents Euler-Mascheroni constant.

Proof Using *lemma7* we get

$$\Gamma(1+x) = 1 - \gamma x + o(x)$$
$$\Gamma\left(1+\frac{1}{a_n}\right) = 1 - \frac{\gamma}{a_n} + o\left(\frac{1}{a_n^2}\right)$$

$$\left(1+\left(o\left(\frac{1}{a_n^2}\right)-\frac{\gamma}{a_n}\right)\right)^{a_n} =$$

$$= \left(1+\left(o\left(\frac{1}{a_n^2}\right)-\frac{\gamma}{a_n}\right)\right)^{\frac{a_n\left(o\left(\frac{1}{a_n^2}-\frac{\gamma}{a_n}\right)\right)}{o\left(\frac{1}{a_n^2}\right)-\frac{\gamma}{a_n}}} =$$

$$= \lim_{n\to\infty} \Gamma^{a_n}\left(1+\frac{1}{a_n}\right) = \lim_{n\to\infty}\left(1+\left(o\left(\frac{1}{a_n^2}\right)-\frac{\gamma}{a_n}\right)\right)^{\frac{a_n\left(o\left(\frac{1}{a_n^2}\right)-\frac{\gamma}{a_n}\right)}{o\left(\frac{1}{a_n^2}\right)-\frac{\gamma}{a_n}}} =$$
$$= e^{\lim\left(n\to\infty\, o\left(\frac{1}{a_n}\right)-\gamma\right)} = e^{-\gamma}.$$

4 Conclusion

In this paper, first, briefly are described known historical facts of the gamma function, then previously known facts(lemmas and propositions) are given that are used in the proof of our main results, which have to do with some new relations and properties that apply to the gamma functions, where they are continuation of the papers [4] and [8], estimations are given in the form of infinite series, which include the gamma function, the Euler's number and the Euler-Mascheroni constant [9].

Competing Interests

Authors have declared that no competing interests exist.

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