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Gauss-Mamadu-Njoseh Quadrature Formula for Numerical Integral Interpolation

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

The use of orthogonal polynomials has paved way for researchers to solve complex mathematical formulations expressed in integral and differential forms. In this article, the Gauss-Mamadu-Njoseh quadrature formula is derived for the numerical treatment of integral interpolation. Here, the Mamadu-Njoseh orthogonal polynomials are employed as basis functions to achieve interpolation points for the derived formula. The derived formula offers several advantages such that precision and stability. The method was tested on some selected definite integral equations with numerical evidences showing the effectiveness and accuracy of the derived formula.

Keywords: Mamadu-Njoseh polynomials; orthogonal polynomials; gauss quadrature; definite integral; interpolation.

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1 Introduction

Gaussian quadrature is an approximate technique for evaluating definite integral of the form

$$\int_a^b w(x)g(x)dx \cong \sum_{i=1}^{N} w(x_i)g(x_i),$$

where $w(x)$ is weight function and $g(x)$ is a continuous function. In real life application, the Guassian quadrature is frequently encountered in analyzing approximate methods in statistics. Also, it is used in the computation of mathematical expectations, and to hypergeometric likelihood function estimation [1]. The early mathematical illustrations of seeking the approximate area of small quadrilaterals with irregular shapes birthed the term quadrature [2]. In modern practice of numerical methods, the term quadrature is often viewed from the angle of numerical integration. Carl Friedrich Gauss (1777 -1855) proposed the Gaussian quadrature as it is today from the quadrature rule by the integral of polynomial function to degree of $2m-1$ as sum of m terms [3].

Over the years, integral equations have been coupled with other functional differential equations to model different physical phenomena [4-10]. In many applied problems, the concepts of classical orthogonal polynomials are much very useful. Legendre polynomial was the first known orthogonal polynomials. Then followed by Chebyshev polynomials, Jacobi polynomials, Hermite and Laguerre polynomials [11]. However, the first workable general theory of orthogonal polynomials was first proposed by Chebyshev through a continued expansion of the integral [12]

$$\int_a^b \frac{g(t)}{(x-t)}dt,$$

where $(x-t)$ is convergent forming a system of polynomials that are orthogonal on (a,b) with respect to the weight function g [13].

Over the years, the use of orthogonal polynomials as basis functions in the approximation of many mathematical functions has been on the increase. Consequently, the Gauss quadrature rule has been modified and reformulated into different forms by various researchers through the use orthogonal polynomials as basis functions. The use of Chebyshev, Hermite, Jacobi and Lagrange polynomials as basis functions in the Gauss quadrature results to Chebyshev-Gauss quadrature(CGQ), Gauss-Hermite quadrature (GHQ), Gauss-Jacobi quadrature (GJQ), and Lagrange-Gauss quadrature (LGQ) [14-16]. However, the problems of precision and instability have reduced the efficiency and accuracy among these modified Gauss quadrature rules. Thus, there is need for further reformulation of the Gauss quadrature rule to ensure high degree precision and stability.

In this present study, the Gauss-Mamadu-Njoseh quadrature interpolation (GMNQ) is proposed for the numerical treatment of definite integral. The motivation behind this paper is to ease the limitations of other methods in literature by perturbing the Gaussian qudrature rule by Mamadu-Njose basis functions [17-24] to enhance;

- The identification of relevant interpolating points with ease in the interval [-1,1].
- A systematic procedure for the selection of arbitrary points within the interval [-1,1].
- A suitable means of identification and conversion of limit of integration if not clearly defined in [-1,1].

2 Method of Solution

Let the interval $[a,b]$ consist of an integrable function $g(x)$. If $f(x)$ be a continuously differentiable function in the interval $[a,b]$, and $f(x)=g(x)/w(x)$, where $w(x)$ is a positive integrable function, then we can write

$$\int_a^b g(x)dx = \int_a^b f(x)w(x)dx \tag{2.1}$$

Suppose $(x_i, f(x_i)), i = 0,1,2, \ldots N$, defines $(N+1)$ interpolated points of $f(x)$ for $x_i \in [a,b]$, then we can write

$f(x) \approx \omega_r(x) = \sum_{i=0}^{N} f(x_i)\varphi_i(g(x))$(2.2)

where$\varphi_i(g(x))$ is the Mamadu-Njoseh polynomial of degree Nbeing analytic for all $\omega_r(x)w(x)$.

Suppose we choose x_i for $0 \le i \le N$ as interpolated points for a continuous function $\omega_r(x)/w(x)$, where $\omega_r(x)$ is any rth degree polynomial for $0 \le r \le N$, then by Theorem 2.1, the integration of any function of the form $\omega_r(x)/w(x)$, where in this case $\omega_r(x)$ defines a polynomial of any degree r for $0 \le r \le 2N+1$. Thus, we have,

$$\begin{aligned}\int_a^b g(x)dx &= \int_a^b f(x)w(x)dx \\ &\approx \int_a^b \omega_r(x)w(x)dx \\ &= \sum_{i=0}^{N} f(x_i)\int_a^b \varphi_i\big(g(x)\big)w(x)dx \\ &= \sum_{i=0}^{N} a_i\varphi_i\big(g(x)\big), \qquad (2.2)\end{aligned}$$

where

$$a_i = \int_a^b \varphi_i\big(g(x)\big)w(x)dx. \qquad (2.3)$$

2.1 Special narratives of the Gauss-Mamadu-Njoseh Quadrature formula

In the Gauss-Mamadu-Njoseh formula, $w(x) = 1 + x^2,\ \ x \in [-1,1]$. Hence, $g(x) = f(x)(1 + x^2)$.Let$q_r(x)$ be a polynomial. Then $q_r(x)$ must satisfy the relation

$$\begin{cases}\int_{-1}^{1} q_r(x)w(x)dx = 0, & r \in [N+1, 2N+1], \\ \int_{-1}^{1} q_r(x)w(x)dx = 0, & r = 2N+1\end{cases}.$$

Suppose γ_i be arbitrary interpolated points satisfying $x_i = \gamma_i$. Then, if we integrate in the interval of orthogonality $[-1,1]$ and choose γ_i for $0 \le j \le i$ such that $\varphi_{i+1}(\gamma_i) = 0$ defines the Mamadu-Njoseh polynomials of order $(N+1)$ satisfying all requirements, then we can write

$$q_r(x) = \left(\frac{\varphi_{i+1}(x)}{\beta_{i+1}}\right)^j > 0. \qquad (2.4)$$

If $j = 2,\ r = 2i + 2$, in (2.4), then $\beta_{i+1} = 2^i$ defines all the leading coefficients of $\varphi_{i+1}(x)$.

The error term is defined as

$$e_r(x) = \int_{-1}^{1} f(\gamma_0, \dots, \gamma_i, x_0, .., x_i)q_{2i+2}(x)w(x)dx = 0 \qquad (2.5)$$

Further simplification of (2.5) yields,

$$\begin{aligned}e_r(x) &= \left(\frac{g^{2i+2}(z)}{\beta_{i+1}^2(2i+2)!}\right)\int_{-1}^{1} \varphi_{i+1}^2(1 + x^2)dx,\ z \in [-1,1] \\ &= \left(\frac{g^{2i+2}(z)}{2^{2i}(2i+2)!}\right)\int_{-1}^{1} \varphi_{i+1}^2(1 + x^2)dx,\end{aligned}$$

where,

$$\varphi_{i+1}(x) = \sum_{r=0}^{i+1} c_r^{(i+1)} x^r, \qquad (2.6)$$

subject to the conditions

$$\int_{-1}^{1} \varphi_{i+1}(x)\varphi_{j+1}(x)w(x)dx = 0, i \neq j,$$

$$\varphi_{i+1}(1) = 1.$$

In general, suppose there are $(N+1)$ points in Gauss-Mamadu-Njoseh quadrature formula to integrate $g(x)$, then

$$\int_{-1}^{1} f(x)dx \approx \frac{\pi}{n+1}\left(\sum_{i=0}^{N} f(x_i)w(x)\right), \quad \text{for } x_i \in [0,n]. \tag{2.7}$$

Theorem 2.1. Suppose x_i, $i \in [0,1]$ are chosen as roots of Mamadu-Njoseh polynomial with interval of orthogonality$[-1,1]$ and weight function $w(x) = 1 + x^2$, then (2.2) is analytic for Mamadu-Njoseh polynomials of order $2N + 1$.

Proof. Let g(x) be Mamadu-Njoseh polynomials of order $2N + 1$, and w(x) denote the weight function. Define $f(x) = g(x)/w(x)$, and $(x_i, f(x_i))$, $i = 0(1)N$, be data points. Then, interpolating the function f(x) via Mamadu-Njoseh polynomials will lead us to (2.1). Also, $f(x) - \omega_r(x) = 0$ for $x = x_i$. Hence, $f(x) - \omega_r(x) = \varphi_{i+1}(x)q_i(x)$, where $\varphi_{i+1}(x) = \sum_{r=0}^{i+1} c_r^{(i+1)}x^r$, and $q_i(x)$ is a polynomials of order N. Since $\varphi_{i+1}(x)$ is orthogonal, we have

$$\int_a^b g(x)dx = \int_a^b f(x)w(x)dxb$$

$$= \int_a^b \big(f(x) - \omega_r(x)\big)w(x)dx + \int_a^b \omega_r(x)w(x)dx$$

$$= \int_a^b \varphi_{i+1}(x)q_i(x)dx + \int_a^b \omega_r(x)w(x)dx$$

$$= \int_a^b \omega_r(x)w(x)dx$$

3 Numerical Examples

In this section, we experiment the new derived quadrature formula to solve some definite integrals. Resulting numerical evidences will be compared with the exact solution, and other methods in literature.

Example 3.1: Compute $I = \int_{-1}^{1} e^{-x^2}dx$, correct to 4 decimal places, using Gauss-Mamadu-Njoseh quadrature formula, analytic for $\omega_r(x)w(x)$, where $\omega_r(x)$ is a polynomial and $r \in [0,5]$.

The interpolated points are defined by the zeroes of $\varphi_2(x)$ being that the Gauss-Mamadu-Njoseh quadrature is analytic for all functions of the form $\omega_r(x)w(x)$. Thus, the zeroes of $\varphi_2(x) = \frac{1}{5}(14x^2 - 2x) = 0$ are $x_0 = -\frac{\sqrt{42}}{14}$, $x_1 = 0$ and $x_2 = \frac{\sqrt{42}}{14}$. Hence, using (2.3), we obtain $a_0 = a_1 = a_2 = 1.047197551$. Thus, by (2.2), we obtain

$$\int_{-1}^{1} e^{-x^2}dx \approx \sum_{i=0}^{2} a_i\varphi_i\left(e^{-x_i{}^2}(1 + x_i^2)\right) = 1.4919$$

Computational results are presented in Table 1 below.

Table 1. Computational Results for Example 3.1

Exact	***GMNQ***	***CGQ***	***GHQ***	***LGQ***
1.4936	1.4919	1.4912	1.5317	1.5421
Error	0.0017	0.0024	0.0381	0.0485

Example 3.2: Compute $I = \int_0^{\frac{\pi}{2}} \sin x \, dx$, correct to 4 decimal places, using Gauss-Mamadu-Njoseh quadrature formula, analytic for $\omega_r(x)w(x)$, where $\omega_r(x)$ is a polynomial and $r \in [0,2]$.

Here, the interval of integration $\left[0, \frac{\pi}{2}\right]$ must be converted to $[-1,1]$. Hence, $z \in [-1,1]$ when $x \in \left[0, \frac{\pi}{2}\right]$. Thus, $\varphi_2(z) = \frac{1}{2}(14z^2 - 2z) = 0$ and $w(z) = 1 + z^2$. Hence, $z_0 = -\frac{\sqrt{10}}{5}$, and $z_1 = -\frac{\sqrt{10}}{5}$. Hence, using (2.3), we obtain $a_0 = a_1 = a_2 = 1.570796227$.

$$\int_0^{\frac{\pi}{2}} \sin x \, dx \approx \sum_{i=0}^{2} a_i \varphi_i \left(e^{-x_i^{\,2}}(1 + x_i^2)\right) = 1.04820$$

The computational values obtained from Example 3.2 are presented in Table 2 below.

Table 2. Computational Results

Exact	**GMNQ**	**CGQ**	**GHQ**	**LGQ**
1.04820	1.04621	1.04921	1.04820	1.03621
Error	0.00199	0.00101	0.08801	0.01199

4 Discussion of Results

The Gauss-Mamadu-Njoseh quadrature formula has been derived and implemented for the approximate solutions of definite integrals. The resulting numerical evidences show that the derived formula is equivocally accurate and converges to the analytic solution. For instance, in **Table 1** and **Table 2**, it is seen that the *GMNQ* and *CGQ* converges with a maximum errors of order 10^{-3} and 10^{-3}, respectively. The *GMNQ* and *CGQ* has same interval of orthogonality but different weight functions, which may account for the equal convergence rate. In comparison, the GMNQ and CGQ perform better than GHQ and LGQ, which attained maximum errors of order $10^{-2}$and 10^{-2}, respectively.

5 Conclusion

The Gauss-Mamadu-Njoseh quadrature formula has been derived and implemented for the solution of definite integral equations. The Mamadu-Njoseh polynomials were targeted as interpolation points to ensure precision, stability and accuracy of the entire procedure. From the resulting numerical evidences, we equivocally assert that the Gauss-Mamadu-Njoseh quadrature formula should be viewed in the same light as that of Chebyshev-Gauss quadrature formula.

Competing Interests

Authors have declared that no competing interests exist.

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