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On Orthogonal Double Covers of Complete Bipartite Graphs by an Infinite Certain Graph-Path and Graph-Cycle

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Original Research Article

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Abstract

Let F be a certain graph, the graph F-Path denoted by $\mathbb{P}_{d+1}(F)$ path of length d with $d+1$ vertices (i.e. Every edge of this path is one-to-one corresponding to an isomorphic to the graph F). In the same manner, we define the graph F-Cycle as $\mathbb{C}_d(F)$ cycle on d vertices. In this paper, we construct orthogonal double covers (ODCs) of complete bipartite graph $K_{n,n}$ by $\mathbb{P}_{d+1}(F)$ and $\mathbb{C}_d(F)$.

Keywords: Graph decomposition; Orthogonal double cover; Symmetric starter.

AMS Subject Classification: 05C70, 05B30.

1 Introduction

An ODC of $K_{n,n}$ is a collection $G = \{G_0, G_1, \dots, G_{n-1}, F_0, F_1, \dots, F_{n-1}\}$ of $2n$ subgraphs called pages) of $K_{n,n}$ such that

(i) every edge of $K_{n,n}$ is in exactly one page of $\{G_0, G_1, \dots, G_{n-1}\}$ and in exactly one page of $\{F_0, F_1, \dots, F_{n-1}\}$;

(ii) for $i,\ j \in \{0, 1, \dots, n-1\}$ and $i \neq j,\ E(G_i) \cap E(G_j) = E(F_i) \cap E(F_j) = \emptyset$; and $|E(G_i) \cap E(F_j)| = 1$ for all $i,\ j \in \{0, 1, \dots, n-1\}$.

If all the pages are isomorphic to a given graph G, then G is said to be an ODC of $K_{n,n}$ by G.

The vertices of the partite sets of $K_{n,n}$ are denoted by $\{0_0, 1_0, \dots, (n-1)_0\}$ and $\{0_1, 1_1, \dots, (n-1)_1\}$. The length of an edge x_0y_1 of $K_{n,n}$ is defined to be the difference $y - x$, where $x,\ y \in \mathbb{Z}_n = \{0,\ 1,\ 2,\ \dots,\ n-1\}$. Note that sums and differences are carried over in $\mathbb{Z}_n$ (that is, sums and differences are carried modulo n)

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Let G be a subgraph of $K_{n,n}$ and $a \in \mathbb{Z}_n$, then the a-translate of G, denoted by $G + a$ is the edge-induced subgraph of $K_{n,n}$ induced by $\{(x + a)_0(y + a)_1 : x_0y_1 \in E(G)\}$.

A subgraph G of $K_{n,n}$ is called half-starter if $|E(G)| = n$ and the lengths of all edges in G are mutually different. We denote a half-starter G by the vector $v(G) = (v_0,\ v_1,\ \ldots,\ v_{n-1})$, where $v_0,\ v_1,\ \ldots,\ v_{n-1} \in \mathbb{Z}_n$ and v_i can be obtained from the unique edge $(v_i)_0(v_i + i)_1$ of length i in G.

Two half-starters $v(G) = (v_0,\ v_1,\ \ldots,\ v_{n-1})$ and $v(G') = (u_0,\ u_1,\ \ldots,\ u_{n-1})$ are said to be orthogonal if $\{v_i - u_i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n$.

For a subgraph G of $K_{n,n}$ with n edges, the edge-induced subgraph G_s with $E(G_s) = \{y_0x_1 : x_0y_1 \in E(G)\}$ is called the symmetric graph of G. Following three results were established in [1].

I. If G is a half-starter, then the collection of all translates of G forms an edge-decomposition of $K_{n,n}$.

II. If two half-starter $v(G)$ and $v(F)$ are orthogonal, then the union of the set of translates of G and the set of translates of F forms an ODC of $K_{n,n}$.

III. G is a symmetric starter if and only if

$$\{v_i - v_{n-i} + i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n.$$

In this paper, we use the usual notation: P_{m+1} for the path on $m + 1$ vertices, C_n for the cycle on n vertices, $D \overset{L_v}{\cup} F$ for the union of D and F with all the vertices of the set L_v that belongs to each other. In [2], other definitions not introduced here can be found.

Much of research on this subject focused with the detection of ODCs with pages isomorphic to a given graph G. For a summary of results on ODCs, see [3]. In [3,4], this concept has been generalized to ODC of any graph H by a graph G.

While in principle any regular graph is worth considering (e.g., the remarkable case of hypercubes has been investigated in [4]), the choice of $H = K_{n,n}$ is quite natural, also in view of a technical motivation: ODCs of such graphs are of help in order to construct ODCs of K_n (see [1], p. 48).

An algebraic construction of ODCs via "symmetric starters" has been exploited to get a complete classification of ODCs of $K_{n,n}$ by G for $n \leq 9$: a few exceptions apart, all graphs G are found this way (see Table 1 in [1]). This method has been applied in [1,5] to detect some infinite classes of graphs G for which there are ODCs of $K_{n,n}$ by G. In [6], El-Shanawany et al. studied the orthogonal double covers of $K_{n,n}$ by $P_{m+1} \cup^* S_{n-m}$, where n and m are integers, $2 \leq m \leq 10$, $m \leq n$ and $P_{m+1} \cup^* S_{n-m}$ is a tree obtained from the path P_{m+1} with m edges and a star S_{n-m} with $n - m$ edges by identifying an end-vertex of P_{m+1} with the center of S_{n-m}.

Let F be a certain graph, the graph F-Path denoted by $\mathbb{P}_{d+1}(F)$, is a path of set of vertices $\mathbb{V} = \{V_i : 0 \leq i \leq d\}$ and a set of edges $\mathbb{E} = \{E_i : 0 \leq i \leq d - 1\}$ if and only if there exists the following two bijective mappings:

$1.$ $\varphi : \mathbb{E} \to \mathcal{F}$ defined by $\varphi(E_i) = F_i$, where $\mathcal{F} = \{F_0, F_1, \ldots, F_{d-1}\}$ is a collection of d graphs each one is isomorphic to the graph F.

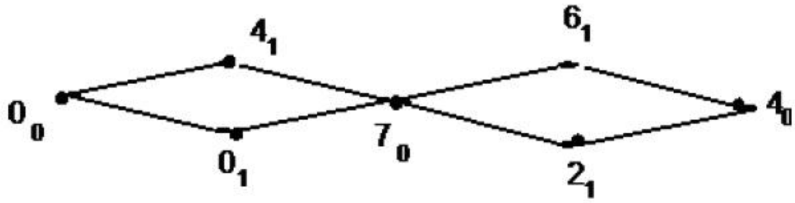

Figure 1: Symmetric starter of an ODC of $K_{8,8}$ by $G = \mathbb{P}_3\left(K_{2,2}\right)$ w.r.t. $\mathbb{Z}_8$

2. $\psi : \mathbb{V} \to \mathcal{A}$ defined by $\psi(V_i) = X_i,$ where $\mathcal{A} = \{X_i : 0 \leq i \leq d : \cap_i X_i = \phi\}$ a class of disjoint sets of vertices.

In the same manner, we define the graph $F-$Cycle as $\mathbb{C}_d\left(F\right)$ cycle on d vertices.

Let $F = K_{r,s}$ be a subgraph of $K_{n,n}$, and for all $0 \leq i \leq d,$

$X_i = \{Y_{\frac{i}{2}} : i \equiv 0 \bmod 2\} \cup \{Z_{\lfloor\frac{i}{2}\rfloor} : i \equiv 1 \bmod 2\}$, satisfying the following conditions:

i. $\mid Y_{\frac{i}{2}} \mid = r$ and $\mid Z_{\lfloor\frac{i}{2}\rfloor} \mid = s,$ for all $0 \leq i \leq d.$

ii. $Y_{\frac{i}{2}} \subset \mathbb{Z}_n \times \{0\}, Z_{\lfloor\frac{i}{2}\rfloor} \subset \mathbb{Z}_n \times \{1\},$ and $\cap_i Y_{\frac{i}{2}} = \cap_i Z_{\lfloor\frac{i}{2}\rfloor} = \phi.$

iii. For $0 \leq i \leq d-1, F_i \cong K_{r,s}$ has the following edges:

$E(F_i) = \{Y_{\frac{i}{2}} Z_{\frac{i}{2}} : i \equiv 0 \bmod 2\} \cup \{Y_{\lfloor\frac{i}{2}\rfloor} Z_{\lceil\frac{i}{2}\rceil} : i \equiv 1 \bmod 2\}.$

Consider $s \geq 0$ paths of length $k \geq 1$, all attached to the same vertex (root vertex). This tree will be called $T\left(s,k\right)$. Clearly, $T\left(s,1\right)$ is the star with s edges and $T\left(1,k\right)$ is the path with k edges. Define L_s is a set of all leaves of $T\left(s,k\right)$ i.e. $L_s = \{v : v$ is a leaf in $T\left(s,k\right)\}$ and $|L_s| = s.$

In this paper, we are concerned with an ODC of $K_{n,n}$ by $G = \mathbb{P}_{d+1}(F),$ Where $F = T\left(s,k\right) \overset{Ls}{\cup} K_1$ and $F = \mathbb{C}_d\left(K_{2,2}\right).$

2 Main Results

El-Shanawany et al. in [7] proved that the vector

$$v\left(G\right) = \left(0, n-1, n-2^2, n-3^2, \cdots, n-3^2, n-2^2, n-1\right) \tag{2.1}$$

is a symmetric starter of an ODC of $K_{n,n}$ by $G,$ where n is a positive integer. Moreover, for any $i \in \mathbb{Z}_n,$ the i^{th} graph isomorphic to G has the edges:

$$E\left(G_i\right) = \left\{\left(n-i^2\right)_0\left(n-i\left(i-1\right)\right)_1 : 0 \leq i \leq n-1\right\} \tag{2.2}$$

As a direct application of the vector in equation (2.1) and its edges in equation (2.2) for special case $n = 2^3$, see figure 1.

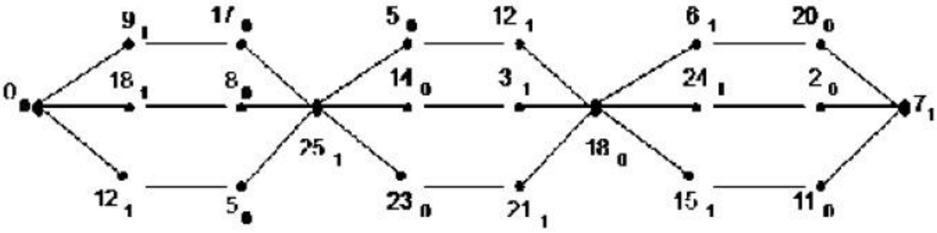

Figure 2: Symmetric starter of an ODC of $K_{27,27}$ by $G = \mathbb{P}_4\left(T\,(3,3) \overset{L_3}{\cup} K_1\right)$ *w.r.t.* $\mathbb{Z}_{27}$

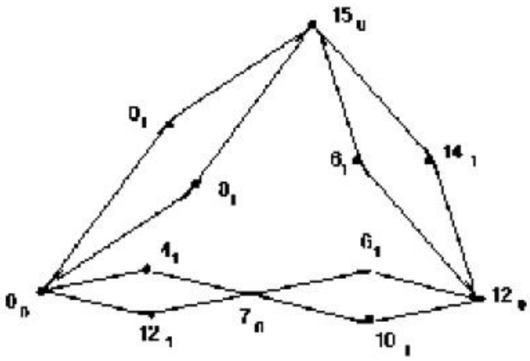

Figure 3: Symmetric starter of an ODC of $K_{16,16}$ by $G = \mathbb{P}_2\,(\mathbb{C}_4\,(K_{2,2})) = \mathbb{C}_4\,(K_{2,2})$ *w.r.t.* $\mathbb{Z}_{16}$

Theorem 2.1. *Let $m \geq 2$ be a positive integer. Then there exists a symmetric starter of an ODC of $K_{3^m,3^m}$ by $G = \mathbb{P}_{3^{m-2}+1}\left(T\,(3,3) \overset{L_3}{\cup} K_1\right)$.*

Proof. The result follows from the vector in equation (2.1) and its edges in equation (2.2) with $|E\,(G)\,| = 3^m = 9d$ imply that $d = 3^{m-2}$ define the number of graphs isomorphic to $F = T\,(3,3) \overset{L_3}{\cup} K_1$. As a direct application of this Theorem, see figure 2. ∎

Theorem 2.2. *Let $m \geq 4$ be a positive integer. Then there exists a symmetric starter of an ODC of $K_{2^m,2^m}$ by $G = \mathbb{P}_{2^{m-4}+1}\,(\mathbb{C}_4\,(K_{2,2}))$.*

Proof. The result follows from the vector in equation (2.1) and its edges in equation (2.2) with $|E\,(G)\,| = 2^m = 16d$ imply that $d = 2^{m-4}$ define the number of graphs isomorphic to $F = \mathbb{C}_4\,(K_{2,2})$ (i.e. the $K_{2,2}$-Cycle with length 4). As a direct applications of this Theorem, see figure 3 and 4. ∎

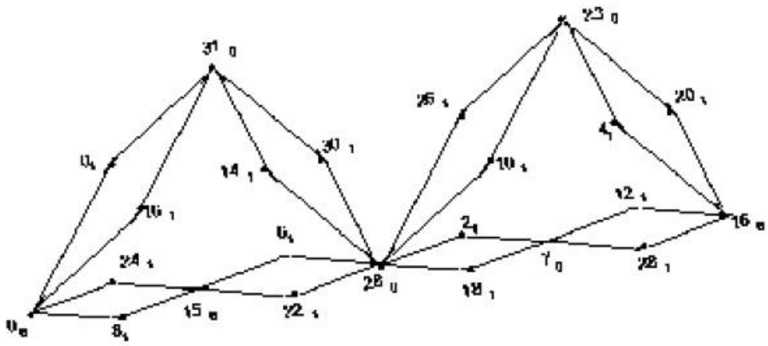

Figure 4: Symmetric starter of an ODC of $K_{32,32}$ by $G = \mathbb{P}_3\left(\mathbb{C}_4\left(K_{2,2}\right)\right)$ *w.r.t.* $\mathbb{Z}_{32}$

3 Conclusions

In this paper we are concerned with a symmetric starter vector of an ODC of $K_{n,n}$ by certain new classes of graphs such as graph-Path and graph-Cycle. In conclusion, we pose the following conjectures:

Conjecture 3.1. *Let n, d, s, k be positive integers. Then there exists a symmetric starter vector of an ODC of $K_{n,n}$ by $G = \mathbb{P}_{d+1}\left(T\left(s,k\right) \overset{L_s}{\cup} K_1\right)$.*

Conjecture 3.2. *Let n, d, r, s be positive integers, then there is a symmetric starter vector of an ODC of the complete bipartite graph $K_{n,n}$ by $G = \mathbb{P}_{d+1}\left(\mathbb{C}_d\ \left(K_{r,s}\right)\right)$.*

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Competing Interests

The author declares that no competing interests exist.

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