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On Some Banach Sequence Spaces Derived by a New Band Matrix

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Abstract

In this paper, we define the band matrix $T = (t_{nk})$ by

$$t_{nk} = \begin{cases} t_n & , \quad k = n \\ -\frac{1}{t_n} & , \quad k = n-1 \\ 0 & , \quad k > n \text{ or } 0 \le k < n-1, \end{cases}$$

where $t_n > 0$ for all $n \in \mathbb{N}$ and $(t_n) \in c \backslash c_0$. By using the matrix T, we introduce the sequence space $\ell_p(T)$ for $1 \le p \le \infty$. Also, we give some inclusion theorems related to this space and find the α-, β-, γ- duals of the space $\ell_p(T)$. Lastly, we investigate the necessary and sufficient conditions for an infinite matrix to be in the classes $(\ell_p(T), \lambda)$ or $(\lambda, \ell_p(T))$ and give the norm of the operators in $B(\ell_p(T), \mu(S))$, where $\lambda \in \{\ell_1, c_0, c, \ell_\infty\}$ and $\mu \in \{\ell_1, \ell_\infty\}$.

Keywords: Sequence spaces; matrix transformations; Schauder basis; $\alpha-, \beta-, \gamma-$duals.
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1 Introduction and Preliminaries

Let ω be the space of all real or complex valued sequences. We shall write $\sup_k$ and $\sum_k$ instead of $\sup_{k\in\mathbb{N}}$ and $\sum_{k=0}^{\infty}$, respectively, where $\mathbb{N} = \{0, 1, 2, ...\}$. Also, if $x = (x_k)_{k=0}^{\infty} \in \omega$, we simply denote it by $x = (x_k)$. Further, $e = (1, 1, ...)$ and $e^{(k)}$ is the sequence whose kth term is 1 and the other terms are 0, that is, $e^{(k)} = (e_0^{(k)}, e_1^{(k)}, ..., e_k^{(k)}, ...) = (0, 0, ..., 1, ...)$. Any vector subspace of ω is called a *sequence space*. By ℓ_∞, c, c_0 and ℓ_p $(1 \le p < \infty)$, we denote the spaces of all bounded, convergent, null sequences and $p-$absolutely convergent series, respectively.

A sequence space λ with a linear topology is called a *K-space* provided each of the maps $p_n : \lambda \to \mathbb{C}$ defined by $p_n(x) = x_n$ is continuous for all $n \in \mathbb{N}$, where $\mathbb{C}$ is the set of all complex numbers. If a K-space λ is a complete linear metric space, it is called an *FK-space*. A normed FK-space is called a *BK-space*, that is, a BK-space is a Banach sequence space. For example, the sequence space ℓ_∞ is a BK-space with the norm given by $\|x\|_{\ell_\infty} = \sup_k |x_k|$. Further, ℓ_p is a complete $p-$normed space and a BK-space in the cases of $0 < p < 1$ and $1 \le p < \infty$ with respect to the usual p-norm and ℓ_p-norm defined by

$$\|x\|_{\ell_p} = \sum_k |x_k|^p \quad (0 < p < 1)$$

and

$$\|x\|_{\ell_p} = \left(\sum_k |x_k|^p\right)^{1/p} \quad (1 \le p < \infty),$$

respectively.

Let λ and μ be sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then A gives a matrix transformation from λ into μ and we write $A : \lambda \to \mu$ if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = (A_n(x))$, the $A-$transform of x, is in μ, where

$$A_n(x) = \sum_k a_{nk}x_k \quad (n \in \mathbb{N}). \tag{1.1}$$

By (λ, μ), we denote the class of all infinite matrices that map λ into μ. Hence $A \in (\lambda, \mu)$ if and only if the series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and $Ax \in \mu$ for all $x \in \lambda$. If λ and μ are any two Banach spaces, then $B(\lambda, \mu)$ denotes the set of all bounded linear operators from λ into μ. The operator norm of $A \in B(\lambda, \mu)$ is given by $\|A\| = \sup\{\|Ax\|_\mu : x \in \lambda, \|x\|_\lambda \le 1\}$.

Let λ be a sequence space. Then the *matrix domain* λ_A of an infinite matrix A is defined by

$$\lambda_A = \{x = (x_k) \in \omega : Ax \in \lambda\}$$

which is also a sequence space.

In the literature, there are many papers related to new sequence spaces constructed by means of the matrix domain of a special triangle. See, for example [1,2,3,4,5,6,7,8]. For more details about matrix domains of triangles, one can see [9].

A sequence (b_n) in normed space λ is called a *Schauder basis* for λ if for every $x \in \lambda$ there is a unique sequence (α_n) of scalars such that $x = \sum_n \alpha_n b_n$, i.e.,

$$\lim_{m\to\infty} \|x - \sum_{n=0}^{m} \alpha_n b_n\| = 0.$$

By cs_0, cs and bs, we denote the set of all convergent to zero, convergent and bounded series, respectively, that is, $cs_0 = \{x = (x_k) \in \omega : \left(\sum_{k=0}^{n} x_k\right)_{n=0}^{\infty} \in c_0\}$, $cs = \{x = (x_k) \in \omega : (\sum_{k=0}^{n} x_k)_{n=0}^{\infty} \in c\}$ and $bs = \{x = (x_k) \in \omega : (\sum_{k=0}^{n} x_k)_{n=0}^{\infty} \in \ell_\infty\}$, and we define the norm on cs_0, cs and bs by $\|x\|_{cs_0} = \|x\|_{cs} = \|x\|_{bs} = \sup_n \left|\sum_{k=0}^{n} x_k\right|$. Let λ and μ be subsets of ω. For all $z \in \omega$, we write $z^{-1} * \mu = \{x \in \omega : xz = (x_k z_k) \in \mu\}$. The set $Z = M(\lambda, \mu) = \cap_{x\in\lambda} x^{-1} * \mu = \{a \in \omega : ax \in \mu \text{ for all } x \in \lambda\}$ is called the multiplier space of λ and μ. In the special case, where $\mu = \ell_1$, $\mu = cs$ or $\mu = bs$, the multiplier spaces $\lambda^\alpha = M(\lambda, \ell_1)$, $\lambda^\beta = M(\lambda, cs)$ and $\lambda^\gamma = M(\lambda, bs)$ are called the α-, β- and γ- *duals* of λ.

Throughout this paper, we assume that $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$.

The difference operator $\Delta : \omega \to \omega$ is defined by $\Delta x = (\Delta x_k) = (x_k - x_{k-1})$ or $\Delta x = (\Delta x_k) = (x_{k-1} - x_k)$ for all $x = (x_k) \in \omega$. The matrix domain λ_Δ is called the difference sequence space whenever λ is a sequence space. Firstly, the notion of difference sequence spaces was defined by Kızmaz [10] as

$$\lambda(\Delta) = \{x = (x_k) \in \omega : (x_{k-1} - x_k) \in \lambda\}$$

for $\lambda = \ell_\infty, c$ and c_0. After Kızmaz [10], Et and Çolak [11] defined the generalized difference sequence spaces

$$\ell_\infty(\Delta^m) = \{x = (x_k) \in \omega : \Delta^m x \in \ell_\infty\},$$

$$c(\Delta^m) = \{x = (x_k) \in \omega : \Delta^m x \in c\}$$

and

$$c_0(\Delta^m) = \{x = (x_k) \in \omega : \Delta^m x \in c_0\},$$

where $m \in \mathbb{N}$, $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ and so that

$$\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}.$$

The difference space

$$bv_p = \{x = (x_k) \in \omega : (x_k - x_{k-1}) \in \ell_p\} \quad (0 < p < \infty)$$

was studied by Altay and Başar [12] for $0 < p < 1$ and in the case $1 \leq p \leq \infty$ Başar and Altay [13], and Çolak et al [14]. Recently, for $\lambda \in \{\ell_p, c_0, c, \ell_\infty\}$ $(1 \leq p < \infty)$, Kirişçi and Başar [6] introduced the generalized difference sequence space

$$\widehat{\lambda} = \{x = (x_k) :\in \omega : B(r, s)x = ((B(r, s)x)_k) \in \lambda\},$$

where $B(r, s)x$ is the sequence defined by $(B(r, s)x)_k = rx_k + sx_{k-1}$ for all $k \in \mathbb{N}$ and $r, s \in \mathbb{R}\backslash\{0\}$. Quite recently, the sequence space

$$\lambda(B) = \{x = (x_k) \in \omega : B(r, s, t)x = ((B(r, s, t)x)_k) \in \lambda\}$$

was studied by Sönmez [15], where $(B(r, s, t)x)_k = rx_k + sx_{k-1} + tx_{k-2}$ for all $k \in \mathbb{N}$ and $r, s, t \in \mathbb{R}\backslash\{0\}$.

In [16], the Fibonacci band matrix $\widehat{F} = (\widehat{f}_{nk})$ was defined by

$$\widehat{f}_{nk} = \begin{cases} -\frac{f_{n+1}}{f_n} & , \quad k = n-1 \\ \frac{f_n}{f_{n+1}} & , \quad k = n \\ 0 & , \quad 0 \leq k < n-1 \text{ or } k > n \end{cases}$$

for all $k, n \in \mathbb{N}$, where f_n is the nth Fibonacci number ($n \in \mathbb{N}$). Also, in [16] the Fibonacci difference sequence spaces introduced as follows:

$$\ell_p(\widehat{F}) = \left\{ x = (x_n) \in \omega : \sum_n \left| \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right|^p < \infty \right\} \quad (1 \leq p < \infty)$$

and

$$\ell_\infty(\widehat{F}) = \left\{ x = (x_n) \in \omega : \sup_n \left| \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right| < \infty \right\},$$

where $\widehat{F}x$ is the $\widehat{F}$-transform of a sequence $x = (x_n)$.

Candan [17] defined the sequential generalized difference matrix $B(\widetilde{r}, \widetilde{s}) = \{b_{nk}(\widetilde{r}, \widetilde{s})\}$ by

$$b_{nk}(\widetilde{r}, \widetilde{s}) = \begin{cases} r_n & , \quad k = n \\ s_n & , \quad k = n-1 \\ 0 & , \quad 0 \leq k < n-1 \text{ or } k > n \end{cases}$$

for all $n, k \in \mathbb{N}$, where $\widetilde{r} = (r_n)$ and $\widetilde{s} = (s_n)$ are convergent sequences of positive real numbers. Moreover, Candan [17] introduced the sequence space

$$\widetilde{\lambda} = \{x = (x_k) \in \omega : B(\widetilde{r}, \widetilde{s})x = ((B(\widetilde{r}, \widetilde{s})x)_k) \in \lambda\},$$

where $(B(\widetilde{r}, \widetilde{s})x)_k = s_{k-1}x_{k-1} + r_k x_k$ for all $k \in \mathbb{N}$, $\lambda \in \{\ell_\infty, c, c_0, \ell_p\}$ and $1 \leq p < \infty$. Further in [18,19,20,21,22,23,24,25], several authors defined and studied some new difference sequence spaces.

The paper is organized so that this section is followed by three sections. In Section 2 we define a new band matrix and introduce the sequence spaces $\ell_p(T)$ and $\ell_\infty(T)$, where $1 \leq p < \infty$. We prove that $\ell_p(T)$ and $\ell_\infty(T)$ are Banach spaces with respect to the norm defined on these spaces. Further, we establish some inclusion theorems related to the space $\ell_p(T)$, where $1 \leq p \leq \infty$. In section 3 we determine the α-, β-, γ- duals of the space $\ell_p(T)$ for $1 \leq p \leq \infty$. In the last section we characterize the classes $(\ell_p(T), \lambda)$ and $(\lambda, \ell_p(T))$ for $\lambda \in \{\ell_1, c_0, c, \ell_\infty\}$ and also give the norm of an operator in the class $(\ell_p(T), \mu(S))$ for $\mu \in \{\ell_1, \ell_\infty\}$, where S is the band matrix having the same properties with T and $1 \leq p \leq \infty$.

2 The Difference Sequence Space $\ell_p(T)$

In the present section, we define a new band matrix T and introduce the difference sequence space $\ell_p(T)$ by using this matrix, where $1 \leq p \leq \infty$. Also, we present some theorems which give inclusion relations corcerning this space.

Let $t_n > 0$ for all $n \in \mathbb{N}$ and $t = (t_n) \in c \backslash c_0$. We define the band matrix $T = (t_{nk})$ by

$$t_{nk} = \begin{cases} t_n & , \quad k = n \\ -\frac{1}{t_n} & , \quad k = n-1 \\ 0 & , \quad k > n \textit{ or } 0 \leq k < n-1. \end{cases}$$

One can easily derive that the inverse $T^{-1} = (g_{nk})$ of the band matrix T is given by

$$g_{nk} = \begin{cases} t_k \prod_{j=k}^{n} \frac{1}{t_j^2} & , \quad 0 \leq k \leq n \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$.

Now, we introduce the difference sequence spaces $\ell_p(T)$ and $\ell_\infty(T)$ by

$$\ell_p(T) = \left\{ x = (x_n) \in \omega : \sum_n \left| t_n x_n - \frac{1}{t_n} x_{n-1} \right|^p < \infty \right\} \quad (1 \leq p < \infty)$$

and

$$\ell_\infty(T) = \left\{ x = (x_n) \in \omega : \sup_n \left| t_n x_n - \frac{1}{t_n} x_{n-1} \right| < \infty \right\}.$$

As the notation of matrix domain, the sequence spaces $\ell_p(T)$ and $\ell_\infty(T)$ may be represented by

$$\ell_p(T) = (\ell_p)_T \quad (1 \leq p < \infty) \quad \text{and} \quad \ell_\infty(T) = (\ell_\infty)_T.$$

The space $\ell_p(T)$ is more general and more comprehensive than the spaces $\ell_p(\widehat{F})$ and bv_p, that is $\ell_p(T)$ contains both of them, where $1 \leq p \leq \infty$. Let $t_n = \frac{f_n}{f_{n+1}}$ $(n \in \mathbb{N})$, then T is the Fibonacci band matrix $\widehat{F}$ and let $(t_n) = e$, then T is the difference matrix Δ. On the other hand, the space $\ell_p(T)$ is not a special case of the space $\widetilde{\ell}_p = (\ell_p)_{B(\widetilde{r},\widetilde{s})}$ defined by Candan [17]. To put it more explicitly, if we take t_k and $-1/t_k$ instead of r_k and s_{k-1}, respectively, this contradicts the fact that $s_k > 0$ for all $k \in \mathbb{N}$.

We will frequently use the sequence $y = (y_n)$ for the T-transform of a sequence $x = (x_n)$, that is,

$$y_n = T_n(x) = \begin{cases} t_0 x_0 & , \quad n = 0 \\ t_n x_n - \frac{1}{t_n} x_{n-1} & , \quad n \geq 1 \end{cases} \quad (n \in \mathbb{N}). \tag{2.1}$$

Theorem 2.1. *Let $1 \leq p \leq \infty$. Then $\ell_p(T)$ is a Banach space with the norm $\|x\|_{\ell_p(T)} = \|Tx\|_{\ell_p}$, that is,*

$$\|x\|_{\ell_p(T)} = \begin{cases} \left(\sum_n |T_n(x)|^p \right)^{1/p} & , \quad 1 \leq p < \infty \\ \sup_n |T_n(x)| & , \quad p = \infty. \end{cases}$$

Proof. Suppose that $\|x\|_{\ell_p(T)} = 0$. Then, $\|Tx\|_{\ell_p} = 0$ and since $\|.\|_{\ell_p}$ is a norm we have $Tx = \theta$. Since T is invertible, we have $x = \theta$.

Let $\alpha \in \mathbb{C}$ and $x \in \ell_p(T)$. Then,

$$\begin{aligned} \|\alpha x\|_{\ell_p(T)} &= \|T(\alpha x)\|_{\ell_p} = \|\alpha T x\|_{\ell_p} \\ &= |\alpha| \|Tx\|_{\ell_p} = |\alpha| \|x\|_{\ell_p(T)}. \end{aligned}$$

Lastly, for $x, z \in \ell_p(T)$ we have

$$\begin{aligned} \|x + z\|_{\ell_p(T)} &= \|T(x + z)\|_{\ell_p} = \|Tx + Tz\|_{\ell_p} \\ &\leq \|Tx\|_{\ell_p} + \|Tz\|_{\ell_p} = \|x\|_{\ell_p(T)} + \|z\|_{\ell_p(T)} \end{aligned}$$

and so the triangle inequality holds.

Hence, $(\ell_p(T), \|.\|_{\ell_p(T)})$ is a normed sequence space for $1 \le p \le \infty$. To prove that $\ell_p(T)$ is a Banach space, let (x_n) be a Cauchy sequece in $\ell_p(T)$. Then, (y_n) is a sequence in ℓ_p. Clearly,

$$\begin{aligned}\|x_n - x_m\|_{\ell_p(T)} &= \|T(x_n - x_m)\|_{\ell_p} \\ &= \|Tx_n - Tx_m\|_{\ell_p} = \|y_n - y_m\|_{\ell_p},\end{aligned}$$

that is, (y_n) is a Cauchy sequence in ℓ_p. Since $(\ell_p, \|.\|_{\ell_p})$ is a Banach space, there exists $y \in \ell_p$ such that $\lim_{n\to\infty} y_n = y$ in ℓ_p. Since $x = T^{-1}y$, we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - x\|_{\ell_p(T)} &= \lim_{n\to\infty} \|T(x_n - x)\|_{\ell_p} \\ &= \lim_{n\to\infty} \|Tx_n - Tx\|_{\ell_p} = \lim_{n\to\infty} \|y_n - y\|_{\ell_p} = 0.\end{aligned}$$

This means that $\lim_{n\to\infty} x_n = x$ in $\ell_p(T)$, where $x \in \ell_p(T)$. The proof is completed.

Remark 2.2. *It is clear that $\ell_p(T)$ is a BK-space for $1 \le p \le \infty$.*

Theorem 2.3. *The difference sequence space $\ell_p(T)$ is linearly isomorphic to the space ℓ_p, that is, $\ell_p(T) \cong \ell_p$ for $1 \le p \le \infty$.*

Proof. We must show that there exists a linear bijection between the spaces $\ell_p(T)$ and ℓ_p for $1 \le p \le \infty$. Let T be the transformation defined from $\ell_p(T)$ to ℓ_p by $x \to y = Tx = (T_n(x))$. Then, we have $Tx = y \in \ell_p$ for every $x \in \ell_p(T)$. It is clear that T is a linear transformation. Also, it is obvious that $x = \theta$ whenever $Tx = \theta$ and so that T is injective.

Furthermore, let $y = (y_n) \in \ell_p$ be given for $1 \le p \le \infty$ and define the sequence $x = (x_n)$ as follows:

$$x_n = \sum_{k=0}^{n} \left(\prod_{j=k}^{n} \frac{1}{t_j^2} \right) t_k y_k \quad (n \in \mathbb{N}). \tag{2.2}$$

Then, by combining (2.1) and (2.2), we get for every $n \in \mathbb{N}$

$$\begin{aligned}T_n(x) &= t_n x_n - \frac{1}{t_n} x_{n-1} \\ &= t_n \sum_{k=0}^{n} \left(\prod_{j=k}^{n} \frac{1}{t_j^2} \right) t_k y_k - \frac{1}{t_n} \sum_{k=0}^{n-1} \left(\prod_{j=k}^{n-1} \frac{1}{t_j^2} \right) t_k y_k \\ &= t_n \left(\prod_{j=n}^{n} \frac{1}{t_j^2} \right) t_n y_n + t_n \sum_{k=0}^{n-1} \left(\prod_{j=k}^{n} \frac{1}{t_j^2} \right) t_k y_k - \frac{1}{t_n} \sum_{k=0}^{n-1} \left(\prod_{j=k}^{n-1} \frac{1}{t_j^2} \right) t_k y_k \\ &= y_n.\end{aligned}$$

This means that $Tx = y$. Since $y \in \ell_p$, we have $Tx \in \ell_p$. Thus, we conclude that $x \in \ell_p(T)$ for any $y \in \ell_p$. Hence T is surjective.

Since $\|x\|_{\ell_p(T)} = \|Tx\|_{\ell_p}$ for any $x \in \ell_p(T)$, we have

$$\|y\|_{\ell_p} = \|Tx\|_{\ell_p} = \|x\|_{\ell_p(T)}$$

which shows that T preserves the norm, where $1 \le p \le \infty$. Hence, T is an isometry. As a consequence, the spaces $\ell_p(T)$ and ℓ_p are isometrically isomorphic for $1 \le p \le \infty$. This completes the proof.

It is known that the space ℓ_p is not a Hilbert space with $p \ne 2$. The similar result is valid for the space $\ell_p(T)$ and the following theorem gives this result.

Theorem 2.4. *The space $\ell_p(T)$ is not an inner product space in the case $p \neq 2$. Hence, $\ell_p(T)$ is not a Hilbert space for $1 \leq p < \infty$ and $p \neq 2$.*

Proof. We must show that the space $\ell_2(T)$ is a Hilbert space while $\ell_p(T)$ is not a Hilbert space for $p \neq 2$. By Theorem 2.1, we know that $\ell_2(T)$ is a Banach space with the norm $\|x\|_{\ell_2(T)} = \|Tx\|_{\ell_2}$ and its norm can be obtained as follows:

$$\|x\|_{\ell_2(T)} = \langle x, x\rangle_{\ell_2(T)}^{1/2} = \langle Tx, Tx\rangle_{\ell_2}^{1/2} = \|Tx\|_{\ell_2}$$

for every $x \in \ell_2(T)$. Hence $\ell_2(T)$ is a Hilbert space.
Consider the sequences

$$u = (u_n) = \begin{cases} \frac{1}{t_0} & , \quad n = 0 \\ (t_1 + \frac{1}{t_0}) \prod_{i=1}^{n} \frac{1}{t_i^2} & , \quad n \geq 1 \end{cases} \quad (n \in \mathbb{N})$$

and

$$v = (v_n) = \begin{cases} \frac{1}{t_0} & , \quad n = 0 \\ (-t_1 + \frac{1}{t_0}) \prod_{i=1}^{n} \frac{1}{t_i^2} & , \quad n \geq 1 \end{cases} \quad (n \in \mathbb{N}).$$

With the T-transforms of u and v, we have the following sequences

$$Tu = (1, 1, 0, 0, ...) \text{ and } Tv = (1, -1, 0, 0, ...).$$

Also, it is easy to see that

$$\|u + v\|_{\ell_p(T)}^2 + \|u - v\|_{\ell_p(T)}^2 = 8 \neq 4(2^{2/p}) = 2(\|u\|_{\ell_p(T)}^2 + \|v\|_{\ell_p(T)}^2)$$

for $p \neq 2$. This means that the parallelogram equality cannot be satisfied by the norm of the space $\ell_p(T)$ for $p \neq 2$. Therefore, this norm cannot be obtained from an inner product. Hence, the space $\ell_p(T)$ with $p \neq 2$ is a Banach space but it is not a Hilbert space, where $1 \leq p < \infty$. The proof is completed.

Remark 2.5. *Obviously, the space $\ell_\infty(T)$ is also a Banach space but it is not a Hilbert space.*

Before giving some inclusion relations concerning the space $\ell_p(T)$, we give a lemma which is necessary to show that some inclusions strictly hold.

Lemma 2.6. *[[26] Theorem 3. page 219]inf.pro] A product $\prod_n(1 + a_n)$ with positive terms a_n is convergent if and only if the series $\sum_n a_n$ converges.*

Now, we present the inclusion relations concerning the space $\ell_p(T)$.

Theorem 2.7. *The inclusion $\ell_p(T) \subset \ell_q(T)$ strictly holds for $1 \leq p < q < \infty$.*

Proof. Let $1 \leq p < q < \infty$. If x is any sequence in $\ell_p(T)$, then its T-transform Tx is in ℓ_p. Since the inclusion $\ell_p \subset \ell_q$ holds, Tx is also in ℓ_q. Hence $x \in \ell_q(T)$ which means that $\ell_p(T) \subset \ell_q(T)$. To show that the inclusion is strict, consider a sequence $x = (x_n) \in \ell_q$ but not in ℓ_p, i.e., $x \in \ell_q \backslash \ell_p$. From the fact that the inclusion $\ell_p \subset \ell_q$ is strict, the existence of $x \in \ell_q \backslash \ell_p$ is clear. Let define the sequence $y = (y_n)$ in terms of the sequence x as follows:

$$y_n = \sum_{k=0}^{n} \left(\prod_{j=k}^{n} \frac{1}{t_j^2} \right) t_k x_k \quad (n \in \mathbb{N}).$$

Then, it is clear that

$$T_n(y) = x_n$$

for every $n \in \mathbb{N}$. This shows that $Ty = x$ and since $x \in \ell_q \backslash \ell_p$, we have $Ty \in \ell_q \backslash \ell_p$. Hence, the sequence y must be in $\ell_q(T)$ but cannot be in $\ell_p(T)$, that is, the inclusion $\ell_p(T) \subset \ell_q(T)$ is strict. The proof is completed.

Theorem 2.8. *The inclusion $\ell_p(T) \subset \ell_\infty(T)$ strictly holds for $1 \le p < \infty$.*

Proof. If $x \in \ell_p(T)$, then $Tx \in \ell_p$. Since $\ell_p \subset \ell_\infty$, $Tx \in \ell_\infty$. Hence, $x \in \ell_\infty(T)$ which shows that $\ell_p(T) \subset \ell_\infty(T)$. To show that this inclusion is strict, we define the sequence $y = (y_n)$ by

$$y_n = \sum_{k=0}^{n} (-1)^k t_k \prod_{i=k}^{n} \frac{1}{t_i^2} \quad (n \in \mathbb{N}).$$

Then, we have for every $n \in \mathbb{N}$ that

$$\begin{aligned} T_n(y) &= t_n \sum_{k=0}^{n} (-1)^k t_k \prod_{i=k}^{n} \frac{1}{t_i^2} - \frac{1}{t_n} \sum_{k=0}^{n-1} (-1)^k t_k \prod_{i=k}^{n-1} \frac{1}{t_i^2} \\ &= t_n (-1)^n t_n \prod_{i=n}^{n} \frac{1}{t_i^2} + t_n \sum_{k=0}^{n-1} (-1)^k t_k \prod_{i=k}^{n} \frac{1}{t_i^2} - \frac{1}{t_n} \sum_{k=0}^{n-1} (-1)^k t_k \prod_{i=k}^{n-1} \frac{1}{t_i^2} \\ &= (-1)^n + \sum_{k=0}^{n-1} (-1)^k t_k \left(t_n \prod_{i=k}^{n} \frac{1}{t_i^2} - \frac{1}{t_n} \prod_{i=k}^{n-1} \frac{1}{t_i^2} \right) \\ &= (-1)^n. \end{aligned}$$

Then, $Ty \in \ell_\infty \backslash \ell_p$ since $((-1)^n) \in \ell_\infty$ but not in ℓ_p. Thus, y is in $\ell_\infty(T)$ but not in $\ell_p(T)$ which means that the inclusion $\ell_p(T) \subset \ell_\infty(T)$ strictly holds. The proof is completed.

Theorem 2.9. *The inclusion $\ell_p \subset \ell_p(T)$ strictly holds, where $1 \le p \le \infty$.*

Proof. To prove that the inclusion $\ell_p \subset \ell_p(T)$ holds for $1 \le p \le \infty$, it is sufficient to show that there exists a number $M > 0$ such that $\|x\|_{\ell_p(T)} \le M \|x\|_{\ell_p}$ for any $x \in \ell_p$.

Let $x \in \ell_p$ and $1 \le p \le \infty$. $\left(\frac{1}{t_n}\right) \in c \backslash c_0$ since $(t_n) \in c \backslash c_0$. Then, there exist $K, L > 0$ such that $t_n \le K$ and $\frac{1}{t_n} \le L$ for all $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned} \|x\|_{\ell_p(T)} &= \left(\sum_n |T_n(x)|^p \right)^{1/p} \\ &= \left(\sum_n \left| t_n x_n - \frac{1}{t_n} x_{n-1} \right|^p \right)^{1/p} \\ &\le \left(\sum_n |t_n x_n|^p \right)^{1/p} + \left(\sum_n \left| \frac{1}{t_n} x_{n-1} \right|^p \right)^{1/p} \\ &\le \left(K^p \sum_n |x_n|^p \right)^{1/p} + \left(L^p \sum_n |x_{n-1}|^p \right)^{1/p} \\ &= (K+L) \|x\|_{\ell_p} \end{aligned}$$

and

$$\begin{aligned} \|x\|_{\ell_\infty(T)} = \sup_n |T_n(x)| &= \sup_n \left| t_n x_n - \frac{1}{t_n} x_{n-1} \right| \\ &\le (K+L) \sup_n |x_n| = (K+L) \|x\|_{\ell_\infty}. \end{aligned}$$

If we define $M = K + L$, it yields the desired result, that is, $\|x\|_{\ell_p(T)} \leq M\|x\|_{\ell_p}$ for $1 \leq p \leq \infty$. To prove that the inclusion is strict:

i) If $0 < t_n < 1$ for all $n \in \mathbb{N}$;

For $1 \leq p < \infty$, the sequence $x = \left(\prod_{i=1}^{n} \frac{1}{t_i^2}\right) \notin \ell_p$ since $1/t_i > 1$. Clearly, $Tx = (t_0, 0, 0, ...) \in \ell_p$. Thus $x \in \ell_p(T)$.

Let $t_i = \sqrt{\frac{i}{i+1}} < 1$ for all $i \in \mathbb{N}$. Then, $\frac{1}{t_i^2} = 1 + \frac{1}{i}$. By Lemma 2.6, $\prod_{i=1}^{\infty} \frac{1}{t_i^2}$ is not convergent since $\sum_{i=1}^{\infty} \frac{1}{i}$ is not convergent. Hence, $x = \left(\prod_{i=1}^{n} \frac{1}{t_i^2}\right)_{n=1}^{\infty} \notin \ell_\infty$. It follows that $Tx = (t_0, 0, 0, ...) \in \ell_\infty$ and so $x \in \ell_\infty(T)$.

ii) If $t_n = 1$ for all $n \in \mathbb{N}$;

In this case,

$$T_n(x) = x_n - x_{n-1}$$

for all $n \in \mathbb{N}$. For $1 \leq p < \infty$, consider the sequence $x = e$ which is clearly not in ℓ_p. But, $Tx = (1, 0, 0, ...) \in \ell_p$, that is, $x \in \ell_p(T)$.

Now, let $x = (n + 1)$. Obviously, x is not in ℓ_∞ but $Tx = e \in \ell_\infty$ which means $x \in \ell_\infty(T)$.

iii) If $t_n > 1$ for all $n \in \mathbb{N}$;

Let $t_n = \frac{n+1}{n} > 1$ for all $n \geq 1$ and assume that $p = 1$. If we choose $x = (\frac{1}{n+1}) \notin \ell_1$, then $Tx = (\frac{1}{n(n+1)}) \in \ell_1$. Hence, $x \in \ell_1(T)$.

Let $t_n = \frac{n+1}{n} > 1$ for all $n \geq 1$ and choose $x = (n)$. Then, $x \notin \ell_\infty$ but $Tx = (\frac{3n+1}{n}) \in \ell_\infty$ and so $x \in \ell_\infty(T)$.

As a result, there exists $x \in \ell_p(T)\backslash\ell_p$. Thus, we conclude that the inclusion $\ell_p \subset \ell_p(T)$ strictly holds for $1 \leq p \leq \infty$. The proof is completed.

Theorem 2.10. *Neither of the spaces ℓ_∞ and $\ell_p(T)$ includes the other one, where $1 \leq p < \infty$.*

Proof. For $t_i = \sqrt{\frac{i}{i+1}}$, consider the sequence $x = \left(\prod_{i=1}^{n} \frac{1}{t_i^2}\right)$ which is not in ℓ_∞. However, $x \in \ell_p(T)$ since $Tx = (t_0, 0, 0, ...) \in \ell_p$. Now, consider the sequence $x = ((-1)^n)$ in ℓ_∞. Thus,

$$T_n(x) = \begin{cases} t_0 & , \quad n = 0 \\ (-1)^n \left(t_n + \frac{1}{t_n}\right) & , \quad n \geq 1 \end{cases} \quad (n \in \mathbb{N}).$$

Clearly, for all $n \in \mathbb{N}$, $\left|(-1)^n \left(t_n + \frac{1}{t_n}\right)\right| > 1$ which implies that the series $\sum_n |T_n(x)|^p$ is not convergent, where $1 \leq p < \infty$. Hence, $x \notin \ell_p(T)$. We conclude that neither of the spaces includes the other.

3 The α-, β- and γ-duals of the Space $\ell_p(T)$

In this section, we determine the α-, β- and γ-duals of the sequence space $\ell_p(T)$, where $1 \leq p \leq \infty$. Also, we give a sequence of the points of the space $\ell_p(T)$ which forms a basis for this space.

The following known results [27] and [28] are fundamental for our investigation.

Lemma 3.1. *Let $1 < p < \infty$. The following statements hold:*

(a) $A = (a_{nk}) \in (\ell_p, \ell_\infty)$ if and only if

$$\sup_n \sum_k |a_{nk}|^q < \infty. \tag{3.1}$$

(b) $A = (a_{nk}) \in (\ell_p, c)$ *if and only if* (3.1) *holds and*

$$\lim_{n\to\infty} a_{nk} \quad exists\ for\ all\ k \in \mathbb{N}. \tag{3.2}$$

(c) $A = (a_{nk}) \in (\ell_p, c_0)$ *if and only if* (3.1) *holds and*

$$\lim_{n\to\infty} a_{nk} = 0\ for\ all\ k \in \mathbb{N}. \tag{3.3}$$

(d) $A = (a_{nk}) \in (\ell_p, \ell_1)$ *if and only if*

$$\sup_{K\in\mathcal{F}} \sum_k \left| \sum_{n\in K} a_{nk} \right|^q < \infty. \tag{3.4}$$

Lemma 3.2. *The following statements hold:*

(a) $A = (a_{nk}) \in (\ell_1, \ell_\infty)$ *if and only if*

$$\sup_{n,k} |a_{nk}| < \infty. \tag{3.5}$$

(b) $A = (a_{nk}) \in (\ell_1, c)$ *if and only if* (3.2) *and* (3.5)
(c) $A = (a_{nk}) \in (\ell_1, c_0)$ *if and only if* (3.3) *and* (3.5)
(d) $A = (a_{nk}) \in (\ell_1, \ell_1)$ *if and only if*

$$\sup_k \sum_n |a_{nk}| < \infty. \tag{3.6}$$

Lemma 3.3. *The following statements hold:*

(a) $A = (a_{nk}) \in (\ell_\infty, \ell_\infty)$ *if and only if* (3.1) *holds with* $q = 1$.
(b) $A = (a_{nk}) \in (\ell_\infty, c)$ *if and only if* (3.2) *holds and*

$$\lim_{n\to\infty} \sum_k |a_{nk}| = \sum_k \left| \lim_{n\to\infty} a_{nk} \right|. \tag{3.7}$$

(c) $A = (a_{nk}) \in (\ell_\infty, c_0)$ *if and only if* (3.3) *holds and*

$$\lim_{n\to\infty} \sum_k |a_{nk}| = 0. \tag{3.8}$$

(d) $A = (a_{nk}) \in (\ell_\infty, \ell_1)$ *if and only if* (3.4) *holds with* $q = 1$.

Now, we give two lemmas which are required to determine the $\alpha-$, β- and γ-duals of the space $\ell_p(T)$, where $1 \le p \le \infty$.

Lemma 3.4. *Let* $a = (a_n) \in \omega$ *and the matrix* $B = (b_{nk})$ *be defined by* $B_n = a_n T_n^{-1}$*, that is,*

$$b_{nk} = \begin{cases} 0 & , \quad k > n \\ a_n g_{nk} & , \quad 0 \le k \le n \end{cases}$$

for all $k, n \in \mathbb{N}$*. Then,* $a \in (\ell_p(T))^\alpha$ *if and only if* $B \in (\ell_p, \ell_1)$*, where* $1 \le p \le \infty$.

Proof. Let y be the T-transform of a sequence $x = (x_n) \in \omega$. Then, we have

$$a_n x_n = a_n T_n^{-1}(y) = B_n(y)$$

for all $n \in \mathbb{N}$. Thus, we obtain from this equality that $ax = (a_n x_n) \in \ell_1$ with $x \in \ell_p(T)$ if and only if $By \in \ell_1$ with $y \in \ell_p$. This implies that $a \in (\ell_p(T))^\alpha$ if and only if $B \in (\ell_p, \ell_1)$. The proof is completed.

Lemma 3.5. *[[29] Theorem 3.1]al-b2] Let $C = (c_{nk})$ be defined via a sequence $a = (a_k) \in \omega$ and the inverse matrix $V = (v_{nk})$ of the triangle matrix $U = (u_{nk})$ by*

$$c_{nk} = \begin{cases} 0 & , \quad k > n \\ \sum_{j=k}^{n} a_j v_{jk} & , \quad 0 \le k \le n \end{cases}$$

for all $k, n \in \mathbb{N}$. Then,

$$(\ell_p(U))^{\gamma} = \{a = (a_k) \in \omega : C \in (\ell_p, \ell_\infty)\},$$
$$(\ell_p(U))^{\beta} = \{a = (a_k) \in \omega : C \in (\ell_p, c)\},$$

where $1 \le p \le \infty$.

Combining Lemmas 3.1-3.5 we have;

Corollary 3.6. *Let the sets $\hat{d}_1, \hat{d}_2, \hat{d}_3, \hat{d}_4, \hat{d}_5$ and $\hat{d}_6$ be defined as follows:*

$$\hat{d}_1 = \left\{ a = (a_k) \in \omega : \sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} \left(t_k \prod_{j=k}^{n} \frac{1}{t_j^2} \right) a_n \right|^q < \infty \right\},$$

$$\hat{d}_2 = \left\{ a = (a_k) \in \omega : \sum_{j=k}^{\infty} \left(t_k \prod_{i=k}^{j} \frac{1}{t_i^2} \right) a_j \text{ exists for each } k \in \mathbb{N} \right\},$$

$$\hat{d}_3 = \left\{ a = (a_k) \in \omega : \sup_n \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \left(t_k \prod_{i=k}^{j} \frac{1}{t_i^2} \right) a_j \right|^q < \infty \right\},$$

$$\hat{d}_4 = \left\{ a = (a_k) \in \omega : \lim_{n \to \infty} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \left(t_k \prod_{i=k}^{j} \frac{1}{t_i^2} \right) a_j \right| = \sum_k \left| \sum_{j=k}^{\infty} \left(\prod_{i=k}^{j} \frac{t_k}{t_i^2} \right) a_j \right| < \infty \right\},$$

$$\hat{d}_5 = \left\{ a = (a_k) \in \omega : \sup_k \sum_n \left| \left(t_k \prod_{j=k}^{n} \frac{1}{t_j^2} \right) a_n \right| < \infty \right\}$$

and

$$\hat{d}_6 = \left\{ a = (a_k) \in \omega : \sup_{n,k} \left| \sum_{j=k}^{n} \left(t_k \prod_{i=k}^{j} \frac{1}{t_i^2} \right) a_j \right| < \infty \right\}.$$

Then, the following statements hold:

(a) $(\ell_p(T))^{\alpha} = \hat{d}_1$ and $(\ell_1(T))^{\alpha} = \hat{d}_5$, where $1 < p \le \infty$.
(b) $(\ell_p(T))^{\beta} = \hat{d}_2 \cap \hat{d}_3$, $(\ell_\infty(T))^{\beta} = \hat{d}_2 \cap \hat{d}_4$ and $(\ell_1(T))^{\beta} = \hat{d}_2 \cap \hat{d}_6$, where $1 < p < \infty$.
(c) $(\ell_p(T))^{\gamma} = \hat{d}_3$ and $(\ell_1(T))^{\gamma} = \hat{d}_6$, where $1 < p \le \infty$.

Now, we give the Schauder basis of the space $\ell_p(T)$ $(1 \le p < \infty)$.

Theorem 3.7. *Let $1 \le p < \infty$ and define the sequence $c^{(k)} \in \ell_p(T)$ for every fixed $k \in \mathbb{N}$ by*

$$(c^{(k)})_n = \begin{cases} 0 & , \quad n < k \\ t_k \prod\limits_{j=k}^{n} \frac{1}{t_j^2} & , \quad n \ge k \end{cases} \quad (n \in \mathbb{N}). \tag{3.9}$$

Then the sequence $(c^{(k)})$ is a basis for the space $\ell_p(T)$, and every $x \in \ell_p(T)$ has a unique representation of the form

$$x = \sum_k T_k(x) c^{(k)}. \tag{3.10}$$

Proof. Let $1 \le p < \infty$. By (3.9), it is clear that $T(c^{(k)}) = e^{(k)} \in \ell_p$ and $c^{(k)} \in \ell_p(T)$ for all $k \in \mathbb{N}$.
Also, let $x \in \ell_p(T)$ given. For every non-negative integer m, we put

$$x^{(m)} = \sum_{k=0}^{m} T_k(x)c^{(k)}.$$

Then, we have that

$$T(x^{(m)}) = \sum_{k=0}^{m} T_k(x)T(c^{(k)}) = \sum_{k=0}^{m} T_k(x)e^{(k)}$$

and hence

$$T_n(x - x^{(m)}) = \begin{cases} 0 & (0 \le n \le m) \\ T_n(x) & (n > m). \end{cases}$$

Let $\epsilon > 0$ be given. Then, there exists a non-negative integer m_0 such that

$$\sum_{n=m_0+1}^{\infty} |T_n(x)|^p \le \left(\frac{\epsilon}{2}\right)^p.$$

Therefore, we obtain for every $m \ge m_0$ that

$$\|x - x^{(m)}\|_{\ell_p(T)} = \left(\sum_{n=m+1}^{\infty} |T_n(x)|^p\right)^{1/p} \le \left(\sum_{n=m_0+1}^{\infty} |T_n(x)|^p\right)^{1/p} \le \frac{\epsilon}{2} < \epsilon$$

which shows that $\lim_{m\to\infty} \|x - x^{(m)}\|_{\ell_p(T)} = 0$ and hence x is represented as in (3.10).

Finally, let us show the uniqueness of the representation (3.10) of $x \in \ell_p(T)$. Assume that $x = \sum_k \mu_k(x)c^{(k)}$. Since the linear transformation T defined from $\ell_p(T)$ to ℓ_p in the proof of Theorem 2.3 is continuous, we have

$$T_n(x) = \sum_k \mu_k(x)T_n(c^{(k)}) = \sum_k \mu_k(x)\delta_{nk} = \mu_n(x) \quad (n \in \mathbb{N}).$$

Hence, the representation (3.10) of $x \in \ell_p(T)$ is unique. The proof is completed.

4 Some Matrix Transformations Related to the Sequence Space $\ell_p(T)$

In the final section, we give the characterization of the classes $(\ell_p(T), \lambda)$, $(\lambda, \ell_p(T))$ and define the norm of an matrix operator in $B(\ell_p(T), \mu(S))$, where $1 \le p \le \infty$, $\lambda \in \{\ell_1, c_0, c, \ell_\infty\}$ and $\mu \in \{\ell_1, \ell_\infty\}$.

Throughout this section, we write $a(n,k) = \sum_{j=0}^{n} a_{jk}$ for given an infinite matrix $A = (a_{nk})$, where $n, k \in \mathbb{N}$.

Firstly, we give a theorem which is essential for our results.

Theorem 4.1. *Let $1 \le p \le \infty$ and λ be an arbitrary subset of ω. Then, we have $A = (a_{nk}) \in (\ell_p(T), \lambda)$ if and only if*

$$D^{(m)} = \left(d_{nk}^{(m)}\right) \in (\ell_p, c) \ for\ all\ n \in \mathbb{N}, \tag{4.1}$$

$$D = (d_{nk}) \in (\ell_p, \lambda), \tag{4.2}$$

where $d_{nk}^{(m)} = \begin{cases} 0 & , \quad k > m \\ \sum_{j=k}^{m} \left(t_k \prod_{i=k}^{j} \frac{1}{t_i^2} \right) a_{nj} & , \quad 0 \le k \le m \end{cases}$ *and* $d_{nk} = \sum_{j=k}^{\infty} \left(t_k \prod_{i=k}^{j} \frac{1}{t_i^2} \right) a_{nj}$ *for all* $k, m, n \in \mathbb{N}$.

Proof. To prove the theorem, we follow the similar way due to Kirişçi and Başar (6). Let $A = (a_{nk}) \in (\ell_p(T), \lambda)$ and $x = (x_k) \in \ell_p(T)$. By (2.2), we have

$$\begin{aligned} \sum_{k=0}^{m} a_{nk} x_k &= \sum_{k=0}^{m} a_{nk} \sum_{j=0}^{k} \left(t_j \prod_{i=j}^{k} \frac{1}{t_i^2} \right) y_j \\ &= \sum_{k=0}^{m} \sum_{j=k}^{m} \left(t_k \prod_{i=k}^{j} \frac{1}{t_i^2} \right) a_{nj} y_k \\ &= \sum_{k=0}^{m} d_{nk}^{(m)} y_k \\ &= D_n^{(m)}(y) \end{aligned}$$

for all $m, n \in \mathbb{N}$. Since Ax exists, $D^{(m)}$ belongs to the class (ℓ_p, c). Letting $m \to \infty$ in the last equality, we obtain $Ax = Dy$ which gives the result $D \in (\ell_p, \lambda)$.

Conversely, suppose the conditions (4.1), (4.2) hold and take any $x \in \ell_p(T)$. Then, we have $(d_{nk})_{k \in \mathbb{N}} \in \ell_p^{\beta}$ which gives together with (4.1) that $A_n = (a_{nk})_{k \in \mathbb{N}} \in (\ell_p(T))^{\beta}$ for all $n \in \mathbb{N}$. Thus, Ax exists. Therefore, we derive by the above equality as $m \to \infty$ that $Ax = Dy$, and this shows that $A \in (\ell_p(T), \lambda)$.

We obtain the following results by combining Lemma 4.1 with Lemmas 3.1-3.3.

Theorem 4.2.

(a) $A = (a_{nk}) \in (\ell_1(T), \ell_\infty)$ *if and only if* (3.5) *holds with* d_{nk} *instead of* a_{nk} *and*

$$\lim_{m \to \infty} d_{nk}^{(m)} \text{ exists } (\forall n, k \in \mathbb{N}), \tag{4.3}$$

$$\sup_{m,k} \left| d_{nk}^{(m)} \right| < \infty \quad (\forall n \in \mathbb{N}) \tag{4.4}$$

also hold.

(b) $A = (a_{nk}) \in (\ell_1(T), c)$ *if and only if* (4.3) *and* (4.4) *hold, and* (3.2) *and* (3.5) *also hold with* d_{nk} *instead of* a_{nk}.

(c) $A = (a_{nk}) \in (\ell_1(T), c_0)$ *if and only if* (4.3) *and* (4.4) *hold, and* (3.3) *and* (3.5) *also hold with* d_{nk} *instead of* a_{nk}.

(d) $A = (a_{nk}) \in (\ell_1(T), \ell_1)$ *if and only if* (4.3) *and* (4.4) *hold, and* (3.6) *also holds with* d_{nk} *instead of* a_{nk}.

Theorem 4.3. *Let* $1 < p < \infty$.

(a) $A = (a_{nk}) \in (\ell_p(T), \ell_\infty)$ *if and only if* (4.3) *and*

$$\sup_{m} \sum_{k=0}^{m} \left| d_{nk}^{(m)} \right|^q < \infty \tag{4.5}$$

hold, and (3.1) *also hold with* d_{nk} *instead of* a_{nk}.

(b) $A = (a_{nk}) \in (\ell_p(T), c)$ *if and only if* (4.3) *and* (4.5) *hold, and* (3.1) *and* (3.2) *also hold with* d_{nk} *instead of* a_{nk}.

(c) $A = (a_{nk}) \in (\ell_p(T), c_0)$ *if and only if* (4.3) *and* (4.5) *hold, and* (3.1) *and* (3.3) *also hold with* d_{nk} *instead of* a_{nk}*.*

(d) $A = (a_{nk}) \in (\ell_p(T), \ell_1)$ *if and only if* (4.3) *and* (4.5) *hold, and* (3.4) *also holds with* d_{nk} *instead of* a_{nk}*.*

Theorem 4.4.

(a) $A = (a_{nk}) \in (\ell_\infty(T), \ell_\infty)$ *if and only if* (4.3) *and*

$$\lim_{m\to\infty} \sum_{k=0}^{m} \left| d_{nk}^{(m)} \right| = \sum_k |d_{nk}| \quad for\ each\ n \in \mathbb{N} \tag{4.6}$$

hold, and (3.1) *also holds with* $q = 1$ *and* d_{nk} *instead of* a_{nk} *.*

(b) $A = (a_{nk}) \in (\ell_\infty(T), c)$ *if and only if* (4.3) *and* (4.6) *hold, and* (3.2) *and* (3.7) *also hold with* d_{nk} *instead of* a_{nk}*.*

(c) $A = (a_{nk}) \in (\ell_\infty(T), c_0)$ *if and only if* (4.3) *and* (4.6) *hold, and* (3.3) *and* (3.8) *also hold with* d_{nk} *instead of* a_{nk}*.*

(d) $A = (a_{nk}) \in (\ell_\infty(T), \ell_1)$ *if and only if* (4.3), (4.6) *and*

$$\sup_{N,K\in\mathcal{F}} \left| \sum_{n\in N} \sum_{k\in K} d_{nk} \right| < \infty$$

hold.

By using Theorems 4.2-4.4, we derive the following results:

Corollary 4.5. *The following statements hold:*

(a) $A = (a_{nk}) \in (\ell_1(T), cs_0)$ *if and only if* (3.3), (3.5) *and* (4.3), (4.4) *hold with* $d(n,k)$ *instead of* a_{nk} *and* d_{nk}*, respectively.*

(b) $A = (a_{nk}) \in (\ell_1(T), cs)$ *if and only if* (3.2), (3.5) *and* (4.3), (4.4) *hold with* $d(n,k)$ *instead of* a_{nk} *and* d_{nk}*, respectively.*

(c) $A = (a_{nk}) \in (\ell_1(T), bs)$ *if and only if* (3.5) *and* (4.3), (4.4) *hold with* $d(n,k)$ *instead of* a_{nk} *and* d_{nk}*, respectively.*

Corollary 4.6. *Let* $1 < p < \infty$*. Then, the following statements hold:*

(a) $A = (a_{nk}) \in (\ell_p(T), cs_0)$ *if and only if* (3.1), (3.3) *and* (4.3), (4.5) *hold with* $d(n,k)$ *instead of* a_{nk} *and* d_{nk}*, respectively.*

(b) $A = (a_{nk}) \in (\ell_p(T), cs)$ *if and only if* (3.1), (3.2) *and* (4.3), (4.5) *hold with* $d(n,k)$ *instead of* a_{nk} *and* d_{nk}*, respectively.*

(c) $A = (a_{nk}) \in (\ell_p(T), bs)$ *if and only if* (3.1) *and* (4.3), (4.5) *hold with* $d(n,k)$ *instead of* a_{nk} *and* d_{nk}*, respectively.*

Corollary 4.7. *The following statements hold:*

(a) $A = (a_{nk}) \in (\ell_\infty(T), cs_0)$ *if and only if* (3.3), (3.8) *and* (4.3), (4.6) *hold with* $d(n,k)$ *instead of* a_{nk} *and* d_{nk}*, respectively.*

(b) $A = (a_{nk}) \in (\ell_\infty(T), cs)$ *if and only if* (3.2), (3.7) *and* (4.3), (4.6) *and hold with* $d(n,k)$ *instead of* a_{nk} *and* d_{nk}*, respectively.*

(c) $A = (a_{nk}) \in (\ell_\infty(T), bs)$ *if and only if* (3.1) *with* $q = 1$ *and* (4.3), (4.6) *hold with* $d(n,k)$ *instead of* a_{nk} *and* d_{nk}*, respectively.*

Now, we introduce the matrix transformations from the space $\lambda \in \{\ell_1, c_0, c, \ell_\infty\}$ to $\ell_p(T)$, where $1 \le p \le \infty$. Before this, we give the necessary and sufficient conditions for the matrix transformation A is in (λ, ℓ_p).

Lemma 4.8. *The following statements hold:*

(a) $A \in (\ell_\infty, \ell_p) = (c, \ell_p) = (c_0, \ell_p)$ *if and only if*

$$\sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} a_{nk} \right|^p < \infty, \text{ where } 1 \leq p < \infty. \tag{4.7}$$

(b) $A \in (\ell_\infty, \ell_\infty) = (c, \ell_\infty) = (c_0, \ell_\infty)$ *if and only if*

$$\sup_n \sum_k |a_{nk}| < \infty. \tag{4.8}$$

(c) $A \in (\ell_1, \ell_p)$ *if and only if*

$$\sup_k \sum_n |a_{nk}|^p < \infty, \text{ where } 1 \leq p < \infty. \tag{4.9}$$

When we change the roles of the spaces $\ell_p(T)$ and ℓ_p with λ in Theorem 4.1, we obtain the following theorem.

Theorem 4.9. *Assume that there exists the following relation between the terms of the infinite matrices* $A = (a_{nk})$ *and* $B = (b_{nk})$

$$b_{nk} = -\frac{1}{t_n} a_{n-1,k} + t_n a_{nk} \tag{4.10}$$

for all $k, n \in \mathbb{N}$ *and* λ *be any given sequence space. Then,* $A \in (\lambda, \ell_p(T))$ *if and only if* $B \in (\lambda, \ell_p)$*, where* $1 \leq p \leq \infty$.

Proof. Let $x = (x_k) \in \lambda$. Then, by using the relation (4.10) one can easily obtain the following equality

$$\sum_{k=0}^{m} b_{nk} x_k = \sum_{k=0}^{m} \left(-\frac{1}{t_n} a_{n-1,k} + t_n a_{nk} \right) x_k \text{ for all } m, n \in \mathbb{N}$$

which yields as $m \to \infty$ that $(B_n(x)) = (T_n(Ax))$. Therefore, we conclude that $Ax \in \ell_p(T)$ for $x \in \lambda$ if and only if $Bx \in \ell_p$ for $x \in \lambda$, where $1 \leq p \leq \infty$. The proof is completed.

By combining Lemma 3.2 (a), Lemma 4.8 and Theorem 4.9, we obtain the following results:

Corollary 4.10. *Let the matrices* $A = (a_{nk})$ *and* $B = (b_{nk})$ *be connected by* (4.10)*. Then, we obtain:*

(a) $A = (a_{nk}) \in (\ell_\infty, \ell_1(T)) = (c, \ell_1(T)) = (c_0, \ell_1(T))$ *if and only if* (4.7) *holds with* $p = 1$ *and* b_{nk} *instead of* a_{nk}.

(b) $A = (a_{nk}) \in (\ell_1, \ell_1(T))$ *if and only if* (4.9) *holds with* $p = 1$ *and* b_{nk} *instead of* a_{nk}.

Corollary 4.11. *Let the matrices* $A = (a_{nk})$ *and* $B = (b_{nk})$ *be connected by* (4.10)*. For* $1 < p < \infty$*, we obtain:*

(a) $A = (a_{nk}) \in (\ell_\infty, \ell_p(T)) = (c, \ell_p(T)) = (c_0, \ell_p(T))$ *if and only if* (4.7) *holds with* b_{nk} *instead of* a_{nk}.

(b) $A = (a_{nk}) \in (\ell_1, \ell_p(T))$ *if and only if* (4.9) *holds with* b_{nk} *instead of* a_{nk}.

Corollary 4.12. *Let the matrices $A = (a_{nk})$ and $B = (b_{nk})$ be connected by (4.10). Then, we obtain:*

(a) $A = (a_{nk}) \in (\ell_\infty, \ell_\infty(T)) = (c, \ell_\infty(T)) = (c_0, \ell_\infty(T))$ if and only if (4.8) holds with b_{nk} instead of a_{nk}.

(b) $A = (a_{nk}) \in (\ell_1, \ell_\infty(T))$ if and only if (3.5) holds with b_{nk} instead of a_{nk}.

Now, we investigate the norm of the infinite matrices in the class $B(\ell_p(T), \mu(S))$, where $\mu \in \{\ell_1, \ell_\infty\}$ and $1 \le p \le \infty$. Firstly, we give an essential lemma for our investigation.

Lemma 4.13. *Let $B = (b_{nk})$ be an infinite matrix. Then the following statements hold:*

(a) The norm of B in $B(\ell_p, \ell_\infty)$ is defined by

$$\|B\|_{(\ell_1,\ell_\infty)} = \sup_{n,k} |b_{nk}|$$

and

$$\|B\|_{(\ell_p,\ell_\infty)} = \sup_n \sum_k |b_{nk}|^q \qquad (1 < p \le \infty).$$

(b) The norm of B in $B(\ell_p, \ell_1)$ is defined by

$$\|B\|_{(\ell_1,\ell_1)} = \sup_k \sum_n |b_{nk}|$$

and

$$\|B\|_{(\ell_p,\ell_1)} = \sup_{K\in\mathcal{F}} \sum_k \left| \sum_{n\in K} b_{nk} \right|^q \qquad (1 < p \le \infty).$$

Theorem 4.14. *Let T and S be two band matrices given by the sequences $t = (t_n)$ and $s = (s_n)$, respectively, and $A = (a_{nk})$ be an infinite matrix.*

(a) If $A \in B((\ell_1(T), \ell_\infty(S))$, then

$$M_1 = \sup_{n,k} \left| \sum_{j=k}^{\infty} \left(t_k \prod_{i=k}^{j} \frac{1}{t_i^2} \right) \left(s_n a_{nj} - \frac{1}{s_n} a_{n-1,j} \right) \right|$$

is finite. In this case, $\|A\|_{(\ell_1(T),\ell_\infty(S))} = M_1$.

(b) Let $1 < p \le \infty$. If $A \in B((\ell_p(T), \ell_\infty(S))$, then

$$M_p = \sup_n \sum_k \left| \sum_{j=k}^{\infty} \left(t_k \prod_{i=k}^{j} \frac{1}{t_i^2} \right) \left(s_n a_{nj} - \frac{1}{s_n} a_{n-1,j} \right) \right|^q$$

is finite. In this case, $\|A\|_{(\ell_p(T),\ell_\infty(S))} = M_p$.

(c) If $A \in B((\ell_1(T), \ell_1(S))$, then

$$K_1 = \sup_k \sum_n \left| \sum_{j=k}^{\infty} \left(t_k \prod_{i=k}^{j} \frac{1}{t_i^2} \right) \left(s_n a_{nj} - \frac{1}{s_n} a_{n-1,j} \right) \right|$$

is finite. In this case, $\|A\|_{(\ell_1(T),\ell_1(S))} = K_1$.

(d) Let $1 < p \leq \infty$. If $A \in B((\ell_p(T), \ell_1(S))$, then

$$K_p = \sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} \sum_{j=k}^{\infty} \left(t_k \prod_{i=k}^{j} \frac{1}{t_i^2} \right) \left(s_n a_{nj} - \frac{1}{s_n} a_{n-1,j} \right) \right|^q$$

is finite. In this case, $\|A\|_{(\ell_p(T), \ell_1(S))} = K_p$.

Proof. From Theorem 2.3, we know that T is an isometric isomorphism: $\ell_p(T) \to \ell_p$, where $1 \leq p \leq \infty$. Let $B = SAT^{-1}$. Thus, the following diagram

$$\begin{array}{ccc} \ell_p(T) & \xrightarrow{A} & \mu(S) \\ {\scriptstyle T^{-1}} \uparrow & & \downarrow {\scriptstyle S} \\ \ell_p & \xrightarrow[B=SAT^{-1}]{} & \mu \end{array}$$

shows that $\|A\|_{(\ell_p(T), \mu(S))} = \|B\|_{(\ell_p, \mu)}$, where $\mu \in \{\ell_\infty, \ell_1\}$ and $1 \leq p \leq \infty$. Using Lemma 4.13 (a) and (b), we have that

$$\|B\|_{(\ell_p, \mu)} = \begin{cases} M_p, & \text{if } \mu = \ell_\infty \\ K_p, & \text{if } \mu = \ell_1, \end{cases}$$

where $1 \leq p \leq \infty$.

5 Conclusions

Introducing a new sequence space by means of the matrix domain of a special triangle has been studied by many mathematicians. In this study we introduce some Banach sequence spaces by using a new band matrix. These spaces are more general than some spaces and not a special case of other spaces defined earlier. These type investigations fill some gaps in the literature. The authors can introduce new sequence spaces and results by using similar techniques in this paper.

Competing Interests

The author declares that no competing interests exist.

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