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Generalized Hyers-Ulam Stability for a Mixed Quadratic - Quartic (QQ) Functional Equation in Quasi-Banach Spaces

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Abstract

In this paper we establish the general solutions and investigate the Hyers - Ulam stability of the following functional equation

$$\begin{aligned} & f(3x+2y+z)+f(3x+2y-z)+f(3x-2y+z)+f(3x-2y-z) \\ & = 72[f(x+y)+f(x-y)]+18[f(x+z)+f(x-z)]+8[f(y+z)+f(y-z)] \\ & \quad +24f(2x)+4f(2y)-240f(x)-160f(y)-48f(z) \end{aligned}$$

in quasi-Banach spaces.

Keywords: Hyers-Ulam stability; quadratic mapping; quartic mapping, mixed type functional equation; quasi - Banach space; p - Banach space.

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1 Introduction and Preliminaries

The following question concerning the stability of homomorphisms is studied by S.M. Ulam [1]: Let $(G_1, *)$ be a group and let $(G_2, \odot, d)$ be a metric group with metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x*y), h(x)\odot h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?. In 1941, D.H. Hyers [2] gave an affirmative answer to the question of Ulam for Banach spaces. In 1950T. Aoki [3] was the second author to treat this problem for additive mappings. In 1978, Th.M. Rassias [4] provided a generalized version of Hyers' theorem which permits the Cauchy difference to become unbounded. A generalization of all the above stability results was obtained by P. Găvruţa [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

H.M. Kim [6] solved the general solutions and proved the Hyers-Ulam stability for the mixed type of quartic and quadratic functional equation:

$$\begin{aligned} &f(x+y+z) + f(x+y-z) + f(x-y+z) + f(x-y-z) + 4f(x) + 4f(y) + 4f(z) \\ &\quad = 2f(x+y) + 2f(x-y) + 2f(x+z) + 2f(x-z) + 2f(y+z) + 2f(y-z). \quad (1.1) \end{aligned}$$

Eshaghi Gordji et al.[7] introduced another mixed type of quartic and quadratic functional equation:

$$f(nx+y) + f(nx-y) = n^2 f(x+y) + n^2 f(x-y) + 2n^2(n^2-1)f(x) - 2(n^2-1)f(y) \quad (1.2)$$

for each fixed integer n with $n \neq 0, \pm 1$. They established the general solutions and proved the Hyers-Ulam stability of this equation in quasi-Banach spaces. Also, for the case $n = 2$, they established the general solution and investigated Hyers - Ulam stability for the following equation:

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 2f(2x) - 8f(x) - 6f(y) \quad (1.3)$$

with $f(0) = 0$ in RN-spaces.

Arunkumar and Agilan [8] introduced the following mixed type of quadratic and additive functional equation

$$\begin{aligned} &f(x+2y+3z) + f(x-2y+3z) + f(x+2y-3z) + f(x-2y-3z) \\ &\quad = 4f(x) + 8[f(y) + f(-y)] + 18[f(z) + f(-z)] \quad (1.4) \end{aligned}$$

and they investigated the Hyers-Ulam stability for Eq. (1.6).

Balamurugan et al. [9, 10] introduced the following mixed type of additive-cubic functional equation

$$\begin{aligned} &f(3x+2y+z) + f(3x+2y-z) + f(3x-2y+z) + f(3x-2y-z) \\ &= 24[f(x+y) + f(x-y)] + 6[f(x+z) + f(x-z)] + 16f(2x) - 80f(x) \quad (1.5) \end{aligned}$$

and they investigated the Hyers-Ulam stability for Eq. (1.5).

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [26] and the references cited therein).

In this paper, we deal with the following functional equation deriving from quartic and quadratic mappings:

$$\begin{aligned} &f(3x+2y+z) + f(3x+2y-z) + f(3x-2y+z) + f(3x-2y-z) \\ &= 72[f(x+y) + f(x-y)] + 18[f(x+z) + f(x-z)] + 8[f(y+z) + f(y-z)] \\ &\qquad + 24f(2x) + 4f(2y) - 240f(x) - 160f(y) - 48f(z) \quad (1.6) \end{aligned}$$

in quasi-Banach spaces.

It is easy to see that the mapping $f(x) = ax^4 + bx^2$ is a solution of the functional equation (1.6).

The main purpose of this paper is to establish the general solution of Eq. (1.6) and investigate the Hyers-Ulam stability for Eq. (1.6).

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.1. (See [24], [25]). Let X be a real linear space. A quasi-norm on X is a real-valued function on X satisfying the following:

(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(iii) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|.\|)$ is called a *quasi-normed* space if $\|.\|$ is a quasi-norm on X. The smallest possible K is called the *modulus of concavity* of $\|.\|$. A *quasi-Banach* space is a complete quasi-normed space. A quasi-norm $\|.\|$ is called a p-norm $(0 < p \leq 1)$ if $\|x + y\|^p \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$. In this case, a quasi-Banach space is called a p-Banach space.

2 General Solutions of Eq. (1.6)

Throughout this section, X and Y will be real vector spaces. Before proceeding the proof of Theorem 2.7 which is the main result in this section, we shall need the following lemmas.

Lemma 2.1. *If a mapping $f : X \to Y$ satisfies the functional equation (1.6) for all $x, y, z \in X$, then the mapping $g : X \to Y$ defined by $g(x) = f(2x) - 16f(x)$ for all $x \in X$ is quadratic.*

Proof. Let $f : X \to Y$ satisfy the functional equation (1.6) for all $x, y, z \in X$. Replacing (x, y, z) by $(0, 0, 0)$ in (1.6), we get $f(0) = 0$. Again replacing (x, y, z) by $(0, 0, x)$ in (1.6), we reach $f(-x) = f(x)$ for all $x \in X$. So the mapping f is even. Replacing (x, y, z) by $(0, x, y)$ in (1.6) and using evenness of f , we obtain

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 2f(2x) - 8f(x) - 6f(y) \tag{2.1}$$

for all $x, y \in X$. Replacing y by $2y$ in (2.1) and using evenness of f, we have

$$f(2x + 2y) + f(2x - 2y) = 4f(2y + x) + 4f(2y - x) + 2f(2x) - 8f(x) - 6f(2y) \tag{2.2}$$

for all $x, y \in X$. Interchanging x with y in (2.2) and then using (2.1), we obtain by evenness of f

$$\begin{aligned} f(2x + 2y) + f(2x - 2y) &= 4f(2x + y) + 4f(2x - y) + 2f(2y) - 8f(y) - 6f(2x) \\ &= 16f(x + y) + 16f(x - y) + 2f(2x) + 2f(2y) - 32f(x) - 32f(y) \end{aligned} \tag{2.3}$$

for all $x, y \in X$. By rearranging (2.3), we have

$$[f(2x+2y) - 16f(x+y)] + [f(2x-2y) - 16f(x-y)] = 2[f(2x) - 16f(x)] + 2[f(2y) - 16f(y)] \tag{2.4}$$

for all $x, y \in X$. This means that $g(x + y) + g(x - y) = 2g(x) + 2g(y)$ for all $x, y \in X$. Therefore the mapping $g : X \to Y$ is quadratic. □

Lemma 2.2. *If a mapping $f : X \to Y$ satisfies the functional equation (1.6) for all $x, y, z \in X$, then the mapping $u : X \to Y$ defined by $u(x) = f(4x) - 256f(x)$ for all $x \in X$ is quadratic.*

Proof. The proof is similar to that of Lemma 2.1 by various substitutions. □

Lemma 2.3. *If a mapping $f : X \to Y$ satisfies the functional equation (1.6) for all $x, y, z \in X$, then the mapping $v : X \to Y$ defined by $v(x) = f(3x) - 81f(x)$ for all $x \in X$ is quadratic.*

Proof. The proof is similar to that of Lemma 2.1 by various substitutions. □

Lemma 2.4. *If a mapping $f : X \to Y$ satisfies the functional equation (1.6) for all $x, y, z \in X$, then the mapping $h : X \to Y$ defined by $h(x) = f(2x) - 4f(x)$ for all $x \in X$ is quartic.*

Proof. It is enough to prove

$$h(2x + y) + h(2x - y) = 4h(x + y) + 4h(x - y) + 24h(x) - 6h(y)$$

for all $x, y \in X$. Replacing (x, y) by $(2x, 2y)$ in (2.1), we get

$$f(4x + 2y) + f(4x - 2y) = 4f(2x + 2y) + 4f(2x - 2y) + 2f(4x) - 8f(2x) - 6f(2y) \tag{2.5}$$

for all $x, y \in X$. Since $g(2x) = 4g(x)$ for all $x \in X$ where $g : X \to Y$ is a quadratic function defined above, we have

$$f(4x) = 20f(2x) - 64f(x) \tag{2.6}$$

for all $x \in X$. Hence, it follows from (2.1), (2.5) and (2.6) that

$$\begin{aligned} h(2x + y) + h(2x - y) &= [f(4x + 2y) - 4f(2x + y)] + [f(4x - 2y) - 4f(2x - y)] \\ &= 4[f(2x + 2y) - 4f(x + y)] + 4[f(2x - 2y) - 4f(x - y)] \\ &\quad + 24[f(2x) - 4f(x)] - 6[f(2y) - 4f(y)] \\ &= 4h(x + y) + 4h(x - y) + 24h(x) - 6h(y) \end{aligned}$$

for all $x, y \in X$. Therefore the mapping $h : X \to Y$ is quartic. □

Lemma 2.5. *If a mapping $f : X \to Y$ satisfies the functional equation (1.6) for all $x, y, z \in X$, then the mapping $s : X \to Y$ defined by $s(x) = f(4x) - 16f(x)$ for all $x \in X$ is quartic.*

Proof. The proof is similar to that of Lemma 2.4 by various substitutions. □

Lemma 2.6. *If a mapping $f : X \to Y$ satisfies the functional equation (1.6) for all $x, y, z \in X$, then the mapping $t : X \to Y$ defined by $t(x) = f(3x) - 3f(x)$ for all $x \in X$ is quartic.*

Proof. The proof is similar to that of Lemma 2.4 by various substitutions. □

Theorem 2.7. *A mapping $f : X \to Y$ satisfies the functional equation (1.6) if and only if there exist a unique symmetric multi-additive mapping $D : X \times X \times X \times X \to Y$ and a unique symmetric bi-additive mapping $B : X \times X \to Y$ such that $f(x) = D(x, x, x, x) + B(x, x)$ for all $x \in X$.*

Proof. We first assume that the mapping $f : X \to Y$ satisfies (1.6). Let $g, h : X \to Y$ be the mappings defined by $g(x) = f(2x) - 16f(x)$ and $h(x) = f(2x) - 4f(x)$ for all $x \in X$. Hence by Lemmas 2.1 and 2.4, we achieve that the mappings g and h are quadratic and quartic respectively and $f(x) = \frac{1}{12}h(x) - \frac{1}{12}g(x)$ for all $x \in X$. Therefore, there exist a unique symmetric multi-additive mapping $D : X \times X \times X \times X \to Y$ and a unique symmetric bi-additive mapping $B : X \times X \to Y$ such that $D(x, x, x, x) = \frac{1}{12}h(x)$ and $B(x, x) = -\frac{1}{12}g(x)$ for all $x \in X$(see [11, 26]). So $f(x) = D(x, x, x, x) + B(x, x)$ for all $x \in X$. The proof of the converse is trivial. □

3 Stability of Eq. (1.6) : Quadratic Case

Throughout this section, assume that X is a quasi-normed space with quasi-norm $\|.\|_X$ and that Y is a $p-$Banach space with $p-$norm $\|.\|_Y$. Let K be the modulus of concavity of $\|.\|_Y$.

In this section, using an idea of [5] we prove the stability of functional equation (1.6). For convenience we use the following abbreviation for a given mapping $f : X \to Y$:

$$\begin{aligned} Df(x,y,z) = & f(3x+2y+z) + f(3x+2y-z) + f(3x-2y+z) + f(3x-2y-z) \\ & - 72[f(x+y)+f(x-y)] - 18[f(x+z)+f(x-z)] - 8[f(y+z)+f(y-z)] \\ & - 24f(2x) - 4f(2y) + 240f(x) + 160f(y) + 48f(z) \end{aligned}$$

for all $x, y, z \in X$.

we will use the following lemma in this section.

Lemma 3.1. *[27] Let $0 < p \leq 1$ and let $x_1, x_2, ..., x_n$ be non-negative real numbers. Then*

$$\left(\sum_{i=1}^{n} x_i\right)^p \leq \left(\sum_{i=1}^{n} x_i^p\right) \tag{3.1}$$

Theorem 3.2. *Let $j \in \{-1, 1\}$ and $\psi_b, M_b : X^3 \to [0, \infty)$ be mappings such that*

$$\lim_{n\to\infty} \frac{\psi_b\left(4^{nj}x, 4^{nj}y, 4^{nj}z\right)}{16^{nj}} = 0, \quad \forall x, y, z \in X, \quad \textit{and} \tag{3.2}$$

$$M_b(x,y,z) = \sum_{i=0}^{\infty} \frac{\psi_b^p\left(4^{ij}x, 4^{ij}y, 4^{ij}z\right)}{16^{pij}} < \infty, \quad \forall x, y, z \in X. \tag{3.3}$$

Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|Df(x,y,z)\|_Y \leq \psi_b(x,y,z), \quad \forall x, y, z \in X. \tag{3.4}$$

Then there exists a unique quadratic mapping $B : X \to Y$ such that

$$\|f(2x) - 16f(x) - B(x)\|_Y \leq \frac{K}{16}\left[\tilde{\psi_b}(x)\right]^{\frac{1}{p}} \tag{3.5}$$

for all $x \in X$, where

$$\begin{aligned} \tilde{\psi_b}(x) = & M_b(x,2x,x) + K^p M_b(x,x,x) + \left(\frac{11}{2}\right)^p K^{2p} M_b(x,0,x) \\ & + 20^p K^{3p} M_b(x,0,0) + K^{4p}\left[\left(\frac{1}{2}\right)^p M_b(0,x,0) + \left(\frac{1}{3}\right)^p M_b(0,0,x)\right] \end{aligned}$$

for all $x \in X$.

Proof. Assume that j=1. Replacing (x, y, z) by $(x, 2x, x), (x, x, x), (x, 0, x)$, $(x, 0, 0)$, $(0, x, 0)$ and $(0, 0, x)$ in (3.4), respectively, we get the following inequalities

$$\begin{aligned} & \|f(8x) + f(6x) - 4f(4x) - 80f(3x) + 118f(2x) + 280f(x) - 72f(-x) + f(-2x)\|_Y \\ & \qquad \leq \psi_b(x,2x,x), \quad \forall x \in X. \end{aligned} \tag{3.6}$$

$$\|f(6x) + f(4x) - 125f(2x) + 448f(x)\|_Y \leq \psi_b(x,x,x), \quad \forall x \in X. \tag{3.7}$$

$$\|2f(4x)-40f(2x)+136f(x)-8f(-x)\|_Y \leq \psi_b\left(x,0,x\right), \quad \forall x \in X. \tag{3.8}$$

$$\|4f(3x)-24f(2x)+60f(x)\|_Y \leq \psi_b\left(x,0,0\right), \quad \forall x \in X. \tag{3.9}$$

$$\|-2f(2x)+72f(x)-72f(-x)+2f(-2x)\|_Y \leq \psi_b\left(0,x,0\right), \quad \forall x \in X. \tag{3.10}$$

$$\|24f(x)-24f(-x)\|_Y \leq \psi_b\left(0,0,x\right), \quad \forall x \in X. \tag{3.11}$$

Let $g, \xi_b : X \to Y$ be mappings defined by $g(x) = f(2x) - 16f(x)$, $\forall x \in X$ and

$$\begin{aligned}\xi_b(x) = K[\psi_b(x,2x,x) + K\psi_b(x,x,x) + \left(\frac{11}{2}\right)K^2\psi_b(x,0,x) + 20K^3\psi_b(x,0,0) \\ + \left(\frac{1}{2}\right)K^4\psi_b(0,x,0) + \left(\frac{1}{3}\right)K^4\psi_b(0,0,x)], \quad \forall x \in X.\end{aligned} \tag{3.12}$$

It follows from (3.6) – (3.12) that

$$\|f(8x)-16f(4x)-16f(2x)+256f(x)\|_Y \leq \xi_b(x), \quad \forall x \in X. \tag{3.13}$$

Therefore (3.13) means

$$\|g(4x)-16g(x)\|_Y \leq \xi_b(x), \quad \forall x \in X. \tag{3.14}$$

By Lemma 3.1 and from (3.2) and (3.3) we infer that

$$\sum_{i=0}^{\infty} \frac{\xi_b^p\left(4^i x\right)}{16^{pi}} < \infty, \qquad \lim_{n\to\infty} \frac{\xi_b\left(4^n x\right)}{16^n} = 0, \quad \forall x \in X. \tag{3.15}$$

Replacing x by $4^n x$ in (3.14) and dividing both sides of (3.14) by 16^{n+1}, we get

$$\left\|\frac{1}{16^{n+1}}g(4^{n+1}x) - \frac{1}{16^n}g(4^n x)\right\|_Y \leq \frac{1}{16^{n+1}}\xi_b(4^n x), \quad \forall x \in X. \tag{3.16}$$

for all $x \in X$ and all non-negative integers n. Since Y is a $p-$Banach space, we have

$$\begin{aligned}\left\|\frac{1}{16^{n+1}}g(4^{n+1}x) - \frac{1}{16^m}g(4^m x)\right\|_Y^p &\leq \sum_{i=m}^{n}\left\|\frac{1}{16^{i+1}}g(4^{i+1}x) - \frac{1}{16^i}g(4^i x)\right\|_Y^p \\ &\leq \frac{1}{16^p}\sum_{i=m}^{n}\frac{1}{16^{pi}}\xi_b^p(4^i x), \quad \forall x \in X\end{aligned} \tag{3.17}$$

and all non-negative integers n and m with $n \geq m$. Therefore we conclude from (3.15) and (3.17) that the sequence $\left\{\frac{1}{16^n}g(4^n x)\right\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\left\{\frac{1}{16^n}g(4^n x)\right\}$ converges in Y for all $x \in X$. So one can define the mapping $B : X \to Y$ by

$$B(x) = \lim_{n\to\infty}\frac{g\left(4^n x\right)}{16^n} \tag{3.18}$$

for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (3.17) and applying Lemma 3.1, we get (3.5). Now, we show that B is a quadratic mapping. It follows from (3.15),(3.16) and (3.18) that

$$\begin{aligned}\|B(4x)-16B(x)\|_Y &= \lim_{n\to\infty}\left\|\frac{1}{16^n}g(4^{n+1}x) - \frac{1}{16^{n-1}}g(4^n x)\right\|_Y \\ &= 16\lim_{n\to\infty}\left\|\frac{1}{16^{n+1}}g(4^{n+1}x) - \frac{1}{16^n}g(4^n x)\right\|_Y \leq 16\lim_{n\to\infty}\frac{\xi_b\left(4^n x\right)}{16^n} = 0\end{aligned}$$

for all $x \in X$. So

$$B(4x) = 16B(x) \tag{3.19}$$

for all $x \in X$. On the other hand it follows from (3.2), (3.4) and (3.18) that

$$\begin{aligned}
\left\|DB(x,y,z)\right\|_Y &= \lim_{n\to\infty} \frac{1}{16^n}\left\|Dg(4^n x, 4^n y, 4^n z)\right\|_Y \\
&= \lim_{n\to\infty} \frac{1}{16^n}\left\|Df(2(4^n)x, 2(4^n)y, 2(4^n)z) - 16Df(4^n x, 4^n y, 4^n z)\right\|_Y \\
&\le \lim_{n\to\infty} \frac{K}{16^n}\left\|Df(4^n(2x), 4^n(2y), 4^n(2z))\|_Y + 16\|Df(4^n x, 4^n y, 4^n z)\right\|_Y \\
&\le \lim_{n\to\infty} \frac{K}{16^n}\left[\psi_b\left(4^n(2x), 4^n(2y), 4^n(2z)\right) + 16\psi_b(4^n x, 4^n y, 4^n z)\right] = 0
\end{aligned}$$

for all $x, y, z \in X$. Hence the mapping B satisfies (1.6). So by Lemma 2.2, the mapping $x \mapsto B(4x) - 256B(x)$ is quadratic. Therefore (3.19) implies that the mapping B is quadratic.

To prove the uniqueness of B, let $S : X \to Y$ be another quadratic mapping satisfying (3.5). It follows from (3.2) and (3.3) that

$$\lim_{n\to\infty} \frac{1}{16^{np}} M_b(4^n x, 4^n y, 4^n z) = \lim_{n\to\infty} \sum_{i=n}^{\infty} \frac{1}{16^{ip}} \psi_b^p(4^n x, 4^n y, 4^n z) = 0, \quad \forall x, y, z \in X.$$

Hence $\lim_{n\to\infty} \frac{1}{16^{np}} \tilde{\psi}_b(4^n x) = 0, \quad \forall x \in X$. So it follows from (3.5) and (3.18) that

$$\left\|B(x) - S(x)\right\|_Y^p = \lim_{n\to\infty} \frac{1}{16^{np}} \left\|g(4^n x) - S(4^n x)\right\|_Y^p \le \frac{K^p}{16^p} \lim_{n\to\infty} \tilde{\psi}_b(4^n x) = 0$$

for all $x \in X$. So $B = S$. Hence the theorem holds for $j = 1$.

Now, replacing x by $\frac{x}{4}$ in (3.14), we reach

$$\|g(x) - 16g(\frac{x}{4})\| \le \xi_b(\frac{x}{4}), \quad \forall x \in X. \tag{3.20}$$

By Lemma 3.1 and the equations (3.2) and (3.3), we infer that

$$\sum_{i=0}^{\infty} 16^{pi} \xi_b^p\left(\frac{x}{4^i}\right) < \infty, \qquad \lim_{n\to\infty} 16^n \xi_b\left(\frac{x}{4^n}\right) = 0, \quad \forall x \in X. \tag{3.21}$$

Replacing x by $\frac{x}{4^n}$ in (3.20) and multiplying both sides of (3.20) to 16^n, we get

$$\left\|16^{n+1} g(\frac{x}{4^{n+1}}) - 16^n g(\frac{x}{4^n})\right\|_Y \le 16^n \xi_b(\frac{x}{4^n}) \tag{3.22}$$

for all $x \in X$ and all non-negative integers n. The rest of the proof is similar to that of $j = 1$. Hence for $j = -1$ also the theorem holds. This completes the proof of the theorem. □

The following corollaries are immediate consequence of Theorem 3.2.

Corollary 3.3. *Let ν, r, s and t be nonnegative real numbers such that r, s and t are all $\neq 2$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|Df(x,y,z)\|_Y \le \begin{cases} \nu, & \\ \nu\,\|x\|_X^r, & r > 0, s = 0, t = 0; \\ \nu\,\|y\|_X^s, & r = 0, s > 0, t = 0; \\ \nu\,\|z\|_X^t, & r = 0, s = 0, t > 0; \\ \nu\left\{\|x\|_X^r + \|y\|_X^s + \|z\|_X^t\right\}, & r > 0, s > 0, t > 0; \end{cases} \tag{3.23}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \to Y$ such that

$$\|f(2x) - 16f(x) - B(x)\|_Y \leq \begin{cases} \alpha_b, & \\ \beta_b(x), & r > 0, s = 0, t = 0; \\ \gamma_b(x), & r = 0, s > 0, t = 0; \\ \delta_b(x), & r = 0, s = 0, t > 0; \\ \zeta_b(x), & r > 0, s > 0, t > 0; \end{cases} \tag{3.24}$$

for all $x \in X$, where

$$\begin{aligned}
\alpha_b =& K\nu \left\{ \frac{1 + K^p + \left(\frac{11}{2}\right)^p K^{2p} + 20^p K^{3p} + \left[\left(\frac{1}{2}\right)^p + \left(\frac{1}{3}\right)^p\right] K^{4p}}{|16^p - 1|} \right\}^{\frac{1}{p}}, \\
\beta_b(x) =& K\nu \left(\frac{4^r}{16}\right) \left\{ \frac{1 + K^p + \left(\frac{7}{2}\right)^p K^{2p} + 6^p K^{3p}}{|16^p - 4^{pr}|} \right\}^{\frac{1}{p}} \|x\|_X^r, \\
\gamma_b(x) =& K\nu \left(\frac{4^s}{16}\right) \left\{ \frac{2^{ps} + K^p + \left(\frac{1}{2}\right)^p K^{4p}}{|16^p - 4^{ps}|} \right\}^{\frac{1}{p}} \|x\|_X^s, \\
\delta_b(x) =& K\nu \left(\frac{4^t}{16}\right) \left\{ \frac{1 + K^p + \left(\frac{11}{2}\right)^p K^{2p} + \left(\frac{1}{3}\right)^p K^{4p}}{|16^p - 4^{pt}|} \right\}^{\frac{1}{p}} \|x\|_X^t \quad \text{and} \\
\zeta_b(x) =& \{\beta_b^p(x) + \gamma_b^p(x) + \delta_b^p(x)\}^{\frac{1}{p}} \quad \text{for all } x \in X.
\end{aligned}$$

Corollary 3.4. *Let $\nu \geq 0$ and r, s, t which are all > 0 be real numbers such that $\lambda = r + s + t \neq 2$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|Df(x, y, z)\|_Y \leq \begin{cases} \nu \left\{ \|x\|_X^r \|y\|_X^s \|z\|_X^t \right\} \\ \nu \left\{ \|x\|_X^r \|y\|_X^s \|z\|_X^t + \|x\|_X^\lambda + \|y\|_X^\lambda + \|z\|_X^\lambda \right\} \end{cases} \tag{3.25}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \to Y$ such that

$$\|f(2x) - 16f(x) - B(x)\|_Y \leq \begin{cases} \rho_b(x), \\ \tau_b(x) \end{cases} \tag{3.26}$$

for all $x \in X$, where

$$\begin{aligned}
\rho_b(x) =& K\nu \left(\frac{4^\lambda}{16}\right) \left\{ \frac{2^{ps} + K^p}{|16^p - 4^{p\lambda}|} \right\}^{\frac{1}{p}} \|x\|_X^\lambda \quad \text{and} \\
\tau_b(x) =& K\nu \left(\frac{4^\lambda}{16}\right) \left\{ \frac{\eta_b(x)}{|16^p - 4^{p\lambda}|} \right\}^{\frac{1}{p}} \|x\|_X^\lambda, \quad \forall x \in X,
\end{aligned}$$

where $\eta_b(x) = 2 + 2^{ps} + 2^{p\lambda} + 4K^p + 2\left(\frac{11}{2}\right)^p K^{2p} + 20^p K^{3p} + \left[\left(\frac{1}{2}\right)^p + \left(\frac{1}{3}\right)^p\right] K^{4p}$.

4 Stability of Eq. (1.6): Quartic Case

Theorem 4.1. *Let $j \in \{-1, 1\}$ and $\psi_d, M_d : X^3 \to [0, \infty)$ be mappings such that*

$$\lim_{n\to\infty} \frac{\psi_d\left(4^{nj}x, 4^{nj}y, 4^{nj}z\right)}{256^{nj}} = 0, \quad \forall x, y, z \in X, \quad \text{and} \tag{4.1}$$

$$M_d(x, y, z) = \sum_{i=0}^{\infty} \frac{\psi_d^p\left(4^{ij}x, 4^{ij}y, 4^{ij}z\right)}{256^{pij}} < \infty, \quad \forall x, y, z \in X. \tag{4.2}$$

Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|Df(x, y, z)\|_Y \leq \psi_d\,(x, y, z)\,, \quad \forall x, y, z \in X. \tag{4.3}$$

Then there exists a unique quartic mapping $D : X \to Y$ such that

$$\|f(2x) - 4f(x) - D(x)\|_Y \leq \frac{K}{256}[\tilde{\psi_d}(x)]^{\frac{1}{p}} \tag{4.4}$$

for all $x \in X$, where

$$\tilde{\psi_d}(x) = M_d(x, 2x, x) + K^p M_d(x, x, x) + \left(\frac{1}{2}\right)^p K^{2p} M_d(x, 0, x)$$
$$+ 20^p K^{3p} M_d(x, 0, 0) + \left(\frac{1}{2}\right)^p K^{4p} M_d(0, x, 0) + \left(\frac{5}{3}\right)^p K^{4p} M_d(0, 0, x), \forall x \in X.$$

Proof. Assume that j=1. Similar to the proof of Theorem 3.2, we have

$$\|f(8x) - 4f(4x) - 256f(2x) + 1024f(x)\|_Y \leq \xi_d(x) \tag{4.5}$$

for all $x \in X$, where

$$\xi_d(x) = K[\psi_d(x, 2x, x) + K\psi_d(x, x, x) + \left(\frac{1}{2}\right) K^2 \psi_d(x, 0, x)$$
$$+ 20K^3 \psi_d(x, 0, 0) + \left(\frac{1}{2}\right) K^4 \psi_d(0, x, 0) + \left(\frac{5}{3}\right) K^4 \psi_d(0, 0, x)], \forall x \in X.$$

Let $h : X \to Y$ be a mapping defined by $h(x) = f(2x) - 4f(x)$, then (4.5) means

$$\|h(4x) - 256h(x)\|_Y \leq \xi_d(x), \quad \forall x \in X. \tag{4.6}$$

By Lemma 3.1 and from (4.1) and (4.2) we infer that

$$\sum_{i=0}^{\infty} \frac{\xi_d^p\,(4^i x)}{256^{pi}} < \infty, \qquad \lim_{n\to\infty} \frac{\xi_d\,(4^n x)}{256^n} = 0, \quad \forall x \in X. \tag{4.7}$$

Replacing x by $4^n x$ in (4.6) and dividing both sides of (4.6) by 256^{n+1}, we get

$$\left\|\frac{1}{256^{n+1}} h(4^{n+1}x) - \frac{1}{256^n} h(4^n x)\right\|_Y \leq \frac{1}{256^{n+1}} \xi_d(4^n x) \tag{4.8}$$

for all $x \in X$ and all non-negative integers n. Since Y is a $p-$Banach space, we have

$$\begin{aligned}\left\|\frac{1}{256^{n+1}} h(4^{n+1}x) - \frac{1}{256^m} h(4^m x)\right\|_Y^p &\leq \sum_{i=m}^{n} \left\|\frac{1}{256^{i+1}} h(4^{i+1}x) - \frac{1}{256^i} h(4^i x)\right\|_Y^p \\ &\leq \frac{1}{256^p} \sum_{i=m}^{n} \frac{1}{256^{pi}} \xi_d^p(4^i x)\end{aligned} \tag{4.9}$$

for all $x \in X$ and all non-negative integers n and m with $n \geq m$. Therefore we conclude from (4.7) and (4.9) that the sequence $\left\{\frac{1}{256^n} h(4^n x)\right\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\left\{\frac{1}{256^n}h(4^n x)\right\}$ converges in Y for all $x \in X$. So one can define the mapping $D: X \to Y$ by

$$D(x) = \lim_{n\to\infty} \frac{h(4^n x)}{256^n} \tag{4.10}$$

for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (4.9) and applying Lemma 3.1, we get (4.4). Now, we show that D is a quartic mapping. It follows from (4.7),(4.8) and (4.10) that

$$\begin{aligned}\|D(4x) - 256D(x)\|_Y &= \lim_{n\to\infty} \left\| \frac{1}{256^n}h(4^{n+1}x) - \frac{1}{256^{n-1}}h(4^n x)\right\|_Y \\ &= 256\left\| \frac{1}{256^{n+1}}h(4^{n+1}x) - \frac{1}{256^n}h(4^n x)\right\|_Y \le \lim_{n\to\infty} \frac{\xi_d(4^n x)}{256^n} = 0, \ \forall x \in X.\end{aligned}$$

So $D(4x) = 256D(x)$, $\forall x \in X$. The rest of the proof is similar to the proof of the Theorem 3.2. □

The following corollaries are immediate consequence of Theorem 4.1.

Corollary 4.2. *Let ν, r, s and t be nonnegative real numbers such that r, s and t are all $\neq 4$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.23) for all $x, y, z \in X$. Then there exists a unique quartic mapping $D : X \to Y$ such that*

$$\|f(2x) - 4f(x) - D(x)\|_Y \le \begin{cases} \alpha_d, \\ \beta_d(x), & r > 0, s = 0, t = 0; \\ \gamma_d(x), & r = 0, s > 0, t = 0; \\ \delta_d(x), & r = 0, s = 0, t > 0; \\ \zeta_d(x), & r > 0, s > 0, t > 0; \end{cases} \tag{4.11}$$

for all $x \in X$, where

$$\begin{aligned}\alpha_d &= K\nu \left\{ \frac{1 + K^p + \left(\frac{1}{2}\right)^p K^{2p} + 20^p K^{3p} + \left[\left(\frac{1}{2}\right)^p + \left(\frac{5}{3}\right)^p\right] K^{4p}}{|256^p - 1|} \right\}^{\frac{1}{p}}, \\ \beta_d(x) &= K\nu \left(\frac{4^r}{256}\right) \left\{ \frac{1 + K^p + \left(\frac{1}{2}\right)^p K^{2p} + 20^p K^{3p}}{|256^p - 4^{pr}|} \right\}^{\frac{1}{p}} \|x\|_X^r, \\ \gamma_d(x) &= K\nu \left(\frac{4^s}{256}\right) \left\{ \frac{2^{ps} + K^p + \left(\frac{1}{2}\right)^p K^{4p}}{|256^p - 4^{ps}|} \right\}^{\frac{1}{p}} \|x\|_X^s, \\ \delta_d(x) &= K\nu \left(\frac{4^t}{256}\right) \left\{ \frac{1 + K^p + \left(\frac{1}{2}\right)^p K^{2p} + \left(\frac{5}{3}\right)^p K^{4p}}{|256^p - 4^{pt}|} \right\}^{\frac{1}{p}} \|x\|_X^t \qquad \text{and} \\ \zeta_d(x) &= \{\beta_d^p(x) + \gamma_d^p(x) + \delta_d^p(x)\}^{\frac{1}{p}} \quad \text{for all } x \in X.\end{aligned}$$

Corollary 4.3. *Let $\nu \ge 0$ and r, s, t which are all > 0 be real numbers such that $\lambda = r + s + t \neq 4$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.25) for all $x, y, z \in X$. Then there exists a unique quartic mapping $D : X \to Y$ such that*

$$\|f(2x) - 4f(x) - B(x)\|_Y \le \begin{cases} \rho_d(x), \\ \tau_d(x) & \forall x \in X, \quad \text{where}, \end{cases} \tag{4.12}$$

$$\begin{aligned}\rho_d(x) &= K\nu \left(\frac{4^\lambda}{256}\right) \left\{ \frac{2^{ps} + K^p}{|256^p - 4^{p\lambda}|} \right\}^{\frac{1}{p}} \|x\|_X^\lambda \quad \text{and} \\ \tau_d(x) &= K\nu \left(\frac{4^\lambda}{256}\right) \left\{ \frac{\eta_c(x)}{|256^p - 4^{p\lambda}|} \right\}^{\frac{1}{p}} \|x\|_X^\lambda \quad \text{for all } x \in X,\end{aligned}$$

where $\eta_c(x) = 2 + 2^{ps} + 2^{p\lambda} + 4K^p + 2\left(\frac{1}{2}\right)^p K^{2p} + 20^p K^{3p} + \left[\left(\frac{1}{2}\right)^p + \left(\frac{5}{3}\right)^p\right] K^{4p}$.

5 Stability of Eq. (1.6): Mixed Case

Theorem 5.1. *Let $j \in \{-1, 1\}$ and $\psi, M_b, M_d : X^3 \to [0, \infty)$ be mappings such that*

$$\lim_{n\to\infty} \frac{\psi\left(4^{nj}x, 4^{nj}y, 4^{nj}z\right)}{16^{nj}} = 0 = \lim_{n\to\infty} \frac{\psi\left(4^{nj}x, 4^{nj}y, 4^{nj}z\right)}{256^{nj}}, \tag{5.1}$$

$$M_b(x,y,z) = \sum_{i=0}^{\infty} \frac{\psi^p\left(4^{ij}x, 4^{ij}y, 4^{ij}z\right)}{16^{pij}} < \infty \qquad \textit{and}$$

$$M_d(x,y,z) = \sum_{i=0}^{\infty} \frac{\psi^p\left(4^{ij}x, 4^{ij}y, 4^{ij}z\right)}{256^{pij}} < \infty, \quad \forall x, y, z \in X. \tag{5.2}$$

Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|Df(x,y,z)\|_Y \leq \psi\left(x,y,z\right) \tag{5.3}$$

for all $x, y, z \in X$. Then there exist a unique quadratic mapping $B : X \to Y$ and a unique quartic mapping $D : X \to Y$ such that

$$\|f(x) - B(x) - D(x)\|_Y \leq \frac{K^2}{3072}\left\{[16\tilde{\psi}_b(x)]^{\frac{1}{p}} + [\tilde{\psi}_d(x)]^{\frac{1}{p}}\right\} \tag{5.4}$$

for all $x \in X$, where $\tilde{\psi}_b(x)$ and $\tilde{\psi}_d(x)$ for all $x \in X$ are defined as in Theorems 3.2 and 4.1 respectively.

Proof. Let $j = 1$. By Theorems 3.2 and 4.1, there exist a quadratic mapping $B_0 : X \to Y$ and a quartic mapping $D_0 : X \to Y$ such that

$$\|f(2x) - 16f(x) - B_0(x)\|_Y \leq \frac{K}{16}[\tilde{\psi}_b(x)]^{\frac{1}{p}} \qquad \text{and}$$

$$\|f(2x) - 4f(x) - D_0(x)\|_Y \leq \frac{K}{256}[\tilde{\psi}_d(x)]^{\frac{1}{p}}, \qquad \forall x \in X.$$

Therefore it follows from the last two inequalities that

$$\left\|f(x) + \frac{1}{12}B_0(x) - \frac{1}{12}D_0(x)\right\|_Y \leq \frac{K^2}{3072}\left\{[16\tilde{\psi}_b(x)]^{\frac{1}{p}} + [\tilde{\psi}_d(x)]^{\frac{1}{p}}\right\}, \quad \forall x \in X.$$

So we obtain (5.4) by letting $B(x) = -\frac{1}{12}B_0(x)$ and $D(x) = \frac{1}{12}D_0(x), \forall x \in X$. The rest of the proof is similar to the proof of the Theorem 3.2. □

Theorem 5.2. *Let $j \in \{-1, 1\}$ and $\psi, M_b, M_d : X^3 \to [0, \infty)$ be mappings such that*

$$\lim_{n\to\infty} \frac{\psi\left(4^{nj}x, 4^{nj}y, 4^{nj}z\right)}{16^{nj}} = 0 = \lim_{n\to\infty} 256^{nj}\psi\left(4^{nj}x, 4^{nj}y, 4^{nj}z\right), \tag{5.5}$$

$$M_b(x,y,z) = \sum_{i=0}^{\infty} \frac{\psi^p\left(4^{ij}x, 4^{ij}y, 4^{ij}z\right)}{16^{pij}} < \infty \qquad \textit{and}$$

$$M_d(x,y,z) = \sum_{i=0}^{\infty} 256^{pij}\psi^p\left(4^{ij}x, 4^{ij}y, 4^{ij}z\right) < \infty \qquad \forall x, y, z \in X. \tag{5.6}$$

Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (5.3) for all $x, y, z \in X$. Then there exist a unique quadratic mapping $B : X \to Y$ and a unique quartic mapping $D : X \to Y$

such that

$$\|f(x) - B(x) - D(x)\|_Y \leq \frac{K^2}{3072}\left\{[16\tilde{\psi}_b(x)]^{\frac{1}{p}} + [\tilde{\psi}_d(x)]^{\frac{1}{p}}\right\} \tag{5.7}$$

for all $x \in X$, where $\tilde{\psi}_b(x)$ and $\tilde{\psi}_d(x)$ for all $x \in X$ are defined as in Theorems 3.2 and 4.1 respectively.

Proof. The proof is similar to the proof of Theorem 5.1. □

Corollary 5.3. *Let ν, r, s and t be nonnegative real numbers such that r, s and t are all $\neq 2$ and 4. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.23) for all $x, y, z \in X$. Then there exist a unique quadratic mapping $B : X \to Y$ and a unique quartic mapping $D : X \to Y$ such that*

$$\|f(x) - B(x) - D(x)\|_Y \leq \frac{K}{12}\begin{cases} \alpha_b + \alpha_d, & \\ \beta_b(x) + \beta_d(x), & r > 0, s = 0, t = 0; \\ \gamma_b(x) + \gamma_d(x), & r = 0, s > 0, t = 0; \\ \delta_b(x) + \delta_d(x), & r = 0, s = 0, t > 0; \\ \zeta_b(x) + \zeta_d(x), & r > 0, s > 0, t > 0; \end{cases} \tag{5.8}$$

for all $x \in X$, where $\alpha_b, \alpha_d, \beta_b(x), \beta_d(x), \gamma_b(x), \gamma_d(x), \delta_b(x), \delta_d(x), \zeta_b(x)$ and $\zeta_d(x)$ are defined as in Corollaries 3.3 and 4.2

Corollary 5.4. *Let $\nu \geq 0$ and r, s, t which are all > 0 be real numbers such that $\lambda = r + s + t \neq 2$ and 4. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.25) for all $x, y, z \in X$. Then there exist a unique quadratic mapping $B : X \to Y$ and a unique quartic mapping $D : X \to Y$ such that*

$$\|f(x) - B(x) - D(x)\|_Y \leq \frac{K}{12}\begin{cases} \rho_b(x) + \rho_d(x), \\ \tau_b(x) + \tau_d(x) \end{cases} \tag{5.9}$$

for all $x \in X$, where $\rho_b(x), \rho_d(x), \tau_b(x), \tau_d(x)$ are defined as in Corollaries 3.4 and 4.3

6 Stability of Eq.(1.6) Using Various Substitutions

In this section, the Hyers-Ulam stability of (1.6) using various substitutions is investigated. The proofs of the following theorems and corollaries are similar to that of the Theorems 3.2, 4.1, 5.1 and 5.2 and the corollaries 3.3, 3.4, 4.2, 4.3 and 5.3. Hence the details of the proofs are omitted.

Theorem 6.1. *Let $j \in \{-1, 1\}$ and $\psi_b, M_b : X^3 \to [0, \infty)$ be mappings such that*

$$\lim_{n\to\infty} \frac{\psi_b\left(3^{nj}x, 3^{nj}y, 3^{nj}z\right)}{9^{nj}} = 0, \quad \forall x, y, z \in X \quad \text{and} \tag{6.1}$$

$$M_b(x, y, z) = \sum_{i=0}^{\infty} \frac{\psi_b^p\left(3^{ij}x, 3^{ij}y, 3^{ij}z\right)}{9^{pij}} < \infty \tag{6.2}$$

for all $x, y, z \in X$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.4) for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \to Y$ such that

$$\|f(2x) - 16f(x) - B(x)\|_Y \leq \frac{K}{9}[\tilde{\psi}_b(x)]^{\frac{1}{p}}, \forall x \in X, \quad \text{where,} \tag{6.3}$$

$$\tilde{\psi}_b(x) = M_b(x, x, x) + K^p\left(\frac{1}{2}\right)M_b(x, 0, x) + K^{2p}\left[4^pM_b(x, 0, 0) + \left(\frac{1}{6}\right)^pM_b(0, 0, x)\right] \quad \forall x \in X.$$

Corollary 6.2. *Let ν, r, s and t be nonnegative real numbers such that r, s and t are all $\neq 2$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.23) for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \to Y$ which satisfies the inequality (3.24) for all $x \in X$, where*

$$\alpha_b = K\nu \left\{ \frac{1 + \left(\frac{1}{2}\right)^p K^p + \left[4^p + \left(\frac{1}{6}\right)^p\right] K^{2p}}{|9^p - 1|} \right\}^{\frac{1}{p}}, \beta_b(x) = K\nu \left(\frac{3^r}{9}\right) \left\{ \frac{1 + \left(\frac{1}{2}\right)^p K^p + 4^p K^{2p}}{|9^p - 3^{pr}|} \right\}^{\frac{1}{p}} \|x\|_X^r,$$

$$\gamma_b(x) = K\nu \left(\frac{3^s}{9}\right) \left\{ \frac{1}{|9^p - 3^{ps}|} \right\}^{\frac{1}{p}} \|x\|_X^s, \delta_b(x) = K\nu \left(\frac{3^t}{9}\right) \left\{ \frac{1 + \left(\frac{1}{2}\right)^p K^p + \left(\frac{1}{6}\right)^p K^{2p}}{|9^p - 3^{pt}|} \right\}^{\frac{1}{p}} \|x\|_X^t$$

and $\zeta_b(x) = \{\beta_b^p(x) + \gamma_b^p(x) + \delta_b^p(x)\}^{\frac{1}{p}}$ *for all $x \in X$.*

Corollary 6.3. *Let $\nu \geq 0$ and r, s, t which are all > 0 be real numbers such that $\lambda = r + s + t \neq 2$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.25) for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \to Y$ which which satisfies the inequality (3.26) for all $x \in X$, where*

$$\begin{aligned} \rho_b(x) =& K\nu \left(\frac{3^\lambda}{9}\right) \left\{ \frac{1}{|9^p - 3^{p\lambda}|} \right\}^{\frac{1}{p}} \|x\|_X^\lambda \quad \textit{and} \\ \tau_b(x) =& K\nu \left(\frac{3^\lambda}{9}\right) \left\{ \frac{4 + 2\left(\frac{1}{2}\right)^p K^p + \left[4^p + \left(\frac{1}{6}\right)^p\right] K^{2p}}{|9^p - 3^{p\lambda}|} \right\}^{\frac{1}{p}} \|x\|_X^\lambda \quad \textit{for all } x \in X. \end{aligned}$$

Theorem 6.4. *Let $j \in \{-1, 1\}$ and $\psi_d, M_d : X^3 \to [0, \infty)$ be mappings such that*

$$\lim_{n\to\infty} \frac{\psi_d\left(3^{nj}x, 3^{nj}y, 3^{nj}z\right)}{81^{nj}} = 0, \quad \textit{and} \tag{6.4}$$

$$M_d(x, y, z) = \sum_{i=0}^{\infty} \frac{\psi_d^p\left(3^{ij}x, 3^{ij}y, 3^{ij}z\right)}{81^{pij}} < \infty \tag{6.5}$$

for all $x, y, z \in X$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (4.3) for all $x, y, z \in X$. Then there exists a unique quartic mapping $D : X \to Y$ such that

$$\|f(2x) - 4f(x) - D(x)\|_Y \leq \frac{K}{81}[\tilde{\psi_d}(x)]^{\frac{1}{p}}, \quad \forall x \in X, \quad \textit{where,} \tag{6.6}$$

$$\tilde{\psi_d}(x) = M_d(x, x, x) + \left(\frac{1}{2}\right)^p M_d(x, 0, x)K^p + [M_d(x, 0, 0) + \left(\frac{1}{6}\right)^p M_d(0, 0, x)]K^{2p} \quad \textit{for all } x \in X.$$

Corollary 6.5. *Let ν, r, s and t be nonnegative real numbers such that r, s and t are all $\neq 4$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.23) for all $x, y, z \in X$. Then there exists a unique quartic mapping $D : X \to Y$ which satisfies the inequality (4.11) for all $x \in X$, where*

$$\alpha_d = K\nu \left\{ \frac{1 + \left(\frac{1}{2}\right)^p K^p + [1 + \left(\frac{1}{6}\right)^p] K^{2p}}{|81^p - 1|} \right\}^{\frac{1}{p}}, \quad \beta_d(x) = K\nu \left(\frac{3^r}{81}\right) \left\{ \frac{1 + \left(\frac{1}{2}\right)^p K^p + K^{2p}}{|81^p - 3^{pr}|} \right\}^{\frac{1}{p}} \|x\|_X^r,$$

$$\gamma_d(x) = K\nu \left(\frac{3^s}{81}\right) \left\{ \frac{1}{|81^p - 3^{ps}|} \right\}^{\frac{1}{p}} \|x\|_X^s, \quad \delta_d(x) = K\nu \left(\frac{3^t}{81}\right) \left\{ \frac{1 + \left(\frac{1}{2}\right)^p K^p + \left(\frac{1}{6}\right)^p K^{2p}}{|81^p - 3^{pt}|} \right\}^{\frac{1}{p}} \|x\|_X^t$$

and $\zeta_d(x) = \{\beta_d^p(x) + \gamma_d^p(x) + \delta_d^p(x)\}^{\frac{1}{p}}$ *for all $x \in X$.*

Corollary 6.6. *Let $\nu \geq 0$ and r, s and t which are all > 0 be real numbers such that $\lambda = r+s+t \neq 4$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.25) for all $x, y, z \in X$. Then there exists a unique quartic mapping $D : X \to Y$ which satisfies the inequality (4.12) for all $x \in X$, where*

$$\rho_d(x) = K\nu \left(\frac{3^\lambda}{81}\right) \left\{\frac{1}{|81^p - 3^{p\lambda}|}\right\}^{\frac{1}{p}} \|x\|_X^\lambda \quad \textit{and}$$
$$\tau_d(x) = K\nu \left(\frac{3^\lambda}{81}\right) \left\{\frac{4 + 2\left(\frac{1}{2}\right)^p K^p + K^{2p} + K^{2p}}{|81^p - 3^{p\lambda}|}\right\}^{\frac{1}{p}} \|x\|_X^\lambda \quad \textit{for all } x \in X.$$

Theorem 6.7. *Let $j \in \{-1, 1\}$ and $\psi, M_b, M_d : X^3 \to [0, \infty)$ be mappings such that*

$$\lim_{n\to\infty} \frac{\psi\left(3^{nj}x, 3^{nj}y, 3^{nj}z\right)}{9^{nj}} = 0 = \lim_{n\to\infty} \frac{\psi\left(3^{nj}x, 3^{nj}y, 3^{nj}z\right)}{81^{nj}} \tag{6.7}$$

$$M_b(x, y, z) = \sum_{i=0}^{\infty} \frac{\psi^p\left(3^{ij}x, 3^{ij}y, 3^{ij}z\right)}{9^{pij}} < \infty \qquad \textit{and}$$
$$M_d(x, y, z) = \sum_{i=0}^{\infty} \frac{\psi^p\left(3^{ij}x, 3^{ij}y, 3^{ij}z\right)}{81^{pij}} < \infty, \quad \forall x, y, z \in X. \tag{6.8}$$

Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (5.3) for all $x, y, z \in X$. Then there exist a unique quadratic mapping $B : X \to Y$ and a unique quartic mapping $D : X \to Y$ such that

$$\|f(x) - B(x) - D(x)\|_Y \leq \frac{K^2}{972} \left\{[9\tilde{\psi}_b(x)]^{\frac{1}{p}} + [\tilde{\psi}_d(x)]^{\frac{1}{p}}\right\} \tag{6.9}$$

for all $x \in X$, where $\tilde{\psi}_b(x)$ and $\tilde{\psi}_d(x)$, for all $x \in X$ are defined as in Theorems 6.1 and 6.4 respectively.

Theorem 6.8. *Let $j \in \{-1, 1\}$ and $\psi, M_b, M_d : X^3 \to [0, \infty)$ be mappings such that*

$$\lim_{n\to\infty} \frac{\psi\left(3^{nj}x, 3^{nj}y, 3^{nj}z\right)}{9^{nj}} = 0 = \lim_{n\to\infty} 81^{nj}\psi\left(3^{nj}x, 3^{nj}y, 3^{nj}z\right) \tag{6.10}$$

$$M_b(x, y, z) = \sum_{i=0}^{\infty} \frac{\psi^p\left(3^{nj}x, 3^{nj}y, 3^{nj}z\right)}{9^{pij}} < \infty, \qquad \textit{and}$$
$$M_d(x, y, z) = \sum_{i=0}^{\infty} 81^{pij}\psi^p\left(3^{nj}x, 3^{nj}y, 3^{nj}zz\right) < \infty, \quad \forall x, y, z \in X. \tag{6.11}$$

Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (5.3) for all $x, y, z \in X$. Then there exist a unique quadratic mapping $B : X \to Y$ and a unique quartic mapping $D : X \to Y$ such that

$$\|f(x) - B(x) - D(x)\|_Y \leq \frac{K^2}{972} \left\{[9\tilde{\psi}_b(x)]^{\frac{1}{p}} + [\tilde{\psi}_d(x)]^{\frac{1}{p}}\right\} \tag{6.12}$$

for all $x \in X$, where $\tilde{\psi}_b(x)$ and $\tilde{\psi}_d(x)$ for all $x \in X$ are defined as in Theorems 3.2 and 4.1 respectively.

Corollary 6.9. *Let ν, r, s and t be nonnegative real numbers such that r, s and t are all $\neq 2$ and 4. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.23) for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \to Y$ and a unique quartic mapping $D : X \to Y$ such that they satisfy the inequality (5.8) for all $x \in X$, where $\alpha_b, \alpha_d, \beta_b(x), \beta_d(x), \gamma_b(x), \gamma_d(x), \delta_b(x), \delta_d(x), \zeta_b(x)$ and $\zeta_d(x)$ are defined as in Corollaries 6.2 and 6.5*

Corollary 6.10. *Let $\nu \geq 0$ and r, s, t which are all > 0 be real numbers such that $\lambda = r+s+t \neq 2$ and 4. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.25) for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \to Y$ and a unique quartic mapping $D : X \to Y$ such that they satisfy the inequality (5.9) for all $x \in X$, where $\rho_b(x), \rho_d(x), \tau_b(x), \tau_d(x)$ are defined as in Corollaries 6.3 and 6.6*

Theorem 6.11. *Let $j \in \{-1, 1\}$ and $\psi_b, M_b : X^3 \to [0, \infty)$ be mappings such that*

$$\lim_{n\to\infty} \frac{\psi_b\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right)}{4^{nj}} = 0, \quad \forall x, y, z \in X, \quad \textit{and} \tag{6.13}$$

$$M_b(x, y, z) = \sum_{i=0}^{\infty} \frac{\psi_b^p\left(2^{ij}x, 2^{ij}y, 2^{ij}z\right)}{4^{pij}} < \infty, \quad \forall x, y, z \in X. \tag{6.14}$$

Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.4) for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \to Y$ such that

$$\|f(2x) - 16f(x) - B(x)\|_Y \leq \frac{K}{4}[\tilde{\psi_b}(x)]^{\frac{1}{p}} \tag{6.15}$$

for all $x \in X$, where

$$\tilde{\psi_b}(x) = M_b(x, x, x) + \left(\frac{1}{4}\right)^p K^p M_b(2x, 0, 0) + 3^p K^{2p} M_b(x, 0, x) + K^{2p} M_b(0, 0, x) \quad \textit{for all } x \in X.$$

Corollary 6.12. *Let $\nu \geq 0$ and r, s and t which are all > 0 be real numbers such that r, s and t are all $\neq 2$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.23) for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \to Y$ which satisfies the inequality (3.24) for all $x \in X$, where*

$$\alpha_b = K\nu \left\{ \frac{1 + \left(\frac{1}{4}\right)^p K^p + 3^p K^{2p} + K^{2p}}{|4^p - 1|} \right\}^{\frac{1}{p}}, \beta_b(x) = K\nu \left(\frac{2^r}{4}\right) \left\{ \frac{1 + 2^{pr}\left(\frac{1}{4}\right)^p K^p + 3^p K^{2p}}{|4^p - 2^{pr}|} \right\}^{\frac{1}{p}} \|x\|_X^r,$$

$$\gamma_b(x) = K\nu \left(\frac{2^s}{4}\right) \left\{ \frac{1}{|4^p - 2^{ps}|} \right\}^{\frac{1}{p}} \|x\|_X^s, \delta_b(x) = K\nu \left(\frac{2^t}{4}\right) \left\{ \frac{1 + 3^p K^{2p} + K^{2p}}{|4^p - 2^{pt}|} \right\}^{\frac{1}{p}} \|x\|_X^t \qquad \textit{and}$$

$$\zeta_b(x) = \{\beta_b^p(x) + \gamma_b^p(x) + \delta_b^p(x)\}^{\frac{1}{p}} \qquad \textit{for all } x \in X.$$

Corollary 6.13. *Let $\nu \geq 0$ and r, s, t which are all > 0 be real numbers such that $\lambda = r+s+t \neq 2$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.25) for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \to Y$ which satisfies the inequality (3.26) for all $x \in X$, where*

$$\rho_b(x) = K\nu \left(\frac{2^\lambda}{4}\right) \left\{ \frac{1}{|4^p - 2^{p\lambda}|} \right\}^{\frac{1}{p}} \|x\|_X^\lambda \quad \textit{and}$$

$$\tau_b(x) = K\nu \left(\frac{2^\lambda}{4}\right) \left\{ \frac{4 + 2^{p\lambda}\left(\frac{1}{4}\right)^p K^p + 2 \cdot 3^p K^{2p} + K^{2p}}{|4^p - 2^{p\lambda}|} \right\}^{\frac{1}{p}} \|x\|_X^\lambda, \quad \forall x \in X.$$

Theorem 6.14. *Let $j \in \{-1, 1\}$ and $\psi_d, M_d : X^3 \to [0, \infty)$ be mappings such that*

$$\lim_{n\to\infty} \frac{\psi_d\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right)}{16^{nj}} = 0, \quad \forall x, y, z \in X, \quad \textit{and} \tag{6.16}$$

$$M_d(x, y, z) = \sum_{i=0}^{\infty} \frac{\psi_d^p\left(2^{ij}x, 2^{ij}y, 2^{ij}z\right)}{16^{pij}} < \infty, \quad \forall x, y, z \in X. \tag{6.17}$$

Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (4.3) for all $x, y, z \in X$. Then there exists a unique quartic mapping $D : X \to Y$ such that

$$\|f(2x) - 4f(x) - D(x)\|_Y \leq \frac{K}{16}[\tilde{\psi_d}(x)]^{\frac{1}{p}} \tag{6.18}$$

for all $x \in X$, where

$$\tilde{\psi_d}(x) = M_d(x, x, x) + \left(\frac{1}{4}\right)^p K^p M_d(2x, 0, 0) + 3^p K^{2p} M_d(x, 0, x) + K^{2p} M_d(0, 0, x) \quad \text{for all } x \in X.$$

Corollary 6.15. *Let $\nu \geq 0$ and r, s and t which are all > 0 be real numbers such that r, s and t are all $\neq 4$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.23) for all $x, y, z \in X$. Then there exists a unique quartic mapping $D : X \to Y$ which satisfies the inequality (4.11) for all $x \in X$, where*

$$\alpha_d = K\nu \left\{ \frac{1 + \left(\frac{1}{4}\right)^p K^p + 3^p K^{2p} + K^{2p}}{|16^p - 1|} \right\}^{\frac{1}{p}}, \beta_d(x) = K\nu \left(\frac{2^r}{16}\right) \left\{ \frac{1 + 2^{pr} \left(\frac{1}{4}\right)^p K^p + 3^p K^{2p}}{|16^p - 2^{pr}|} \right\}^{\frac{1}{p}} \|x\|_X^r,$$

$$\gamma_d(x) = K\nu \left(\frac{2^s}{16}\right) \left\{ \frac{1}{|16^p - 2^{ps}|} \right\}^{\frac{1}{p}} \|x\|_X^s, \delta_d(x) = K\nu \left(\frac{2^s}{16}\right) \left\{ \frac{1 + 3^p K^{2p} + K^{2p}}{|16^p - 2^{pt}|} \right\}^{\frac{1}{p}} \|x\|_X^t \quad \text{and}$$

$$\zeta_d(x) = \{\beta_d^p(x) + \gamma_d^p(x) + \delta_d^p(x)\}^{\frac{1}{p}} \quad \text{for all } x \in X.$$

Corollary 6.16. *Let $\nu \geq 0$ and r, s, t which are all > 0 be real numbers such that $\lambda = r + s + t \neq 4$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.25) for all $x, y, z \in X$. Then there exists a unique quartic mapping $D : X \to Y$ which satisfies the inequality (4.12) for all $x \in X$, where*

$$\rho_d(x) = K\nu \left(\frac{2^\lambda}{16}\right) \left\{ \frac{1}{|16^p - 2^{p\lambda}|} \right\}^{\frac{1}{p}} \|x\|_X^\lambda \quad \text{and}$$

$$\tau_d(x) = K\nu \left(\frac{2^\lambda}{16}\right) \left\{ \frac{4 + 2^{p\lambda} \left(\frac{1}{4}\right)^p K^p + 2 \cdot 3^p K^{2p} + K^{2p}}{|16^p - 2^{p\lambda}|} \right\}^{\frac{1}{p}} \|x\|_X^\lambda \quad \forall x \in X.$$

Theorem 6.17. *Let $j \in \{-1, 1\}$ and $\psi, M_b, M_d : X^3 \to [0, \infty)$ be mappings such that*

$$\lim_{n\to\infty} \frac{\psi\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right)}{4^{nj}} = 0 = \lim_{n\to\infty} \frac{\psi\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right)}{16^{nj}}, \forall x, y, z \in X, \tag{6.19}$$

$$M_b(x, y, z) = \sum_{i=0}^{\infty} \frac{\psi^p(2^{nj}x, 2^{nj}y, 2^{nj}z)}{4^{pij}} < \infty \quad \text{and}$$

$$M_d(x, y, z) = \sum_{i=0}^{\infty} \frac{\psi^p(2^{nj}x, 2^{nj}y, 2^{nj}z)}{16^{pij}} < \infty, \quad \forall x, y, z \in X. \tag{6.20}$$

Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (5.3) for all $x, y, z \in X$. Then there exist a unique quadratic mapping $B : X \to Y$ and a unique quartic mapping $D : X \to Y$ such that

$$\|f(x) - B(x) - D(x)\|_Y \leq \frac{K^2}{192} \left\{ [4\tilde{\psi_b}(x)]^{\frac{1}{p}} + [\tilde{\psi_d}(x)]^{\frac{1}{p}} \right\} \tag{6.21}$$

for all $x \in X$, where $\tilde{\psi_b}(x)$ and $\tilde{\psi_d}(x)$ for all $x \in X$ are defined as in Theorems 6.1 and 6.4 respectively.

Theorem 6.18. *Let* $j \in \{-1, 1\}$ *and* $\psi, M_b, M_d : X^3 \to [0, \infty)$ *be mappings such that*

$$\lim_{n\to\infty} \frac{\psi\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right)}{4^{nj}} = 0 = \lim_{n\to\infty} 16^{nj}\psi\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right), \forall x, y, z \in X, \tag{6.22}$$

$$M_b(x, y, z) = \sum_{i=0}^{\infty} \frac{\psi^p\left(2^{ij}x, 2^{ij}y, 2^{ij}z\right)}{4^{pij}} < \infty \qquad \textit{and}$$

$$M_d(x, y, z) = \sum_{i=0}^{\infty} 16^{pij}\psi^p\left(2^{ij}x, 2^{ij}y, 2^{ij}z\right) < \infty, \quad \forall x, y, z \in X. \tag{6.23}$$

Suppose that a mapping $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality* (5.3) *for all* $x, y, z \in X$. *Then there exist a unique quadratic mapping* $B : X \to Y$ *and a unique quartic mapping* $D : X \to Y$ *such that*

$$\|f(x) - B(x) - D(x)\|_Y \le \frac{K^2}{192}\left\{[4\tilde{\psi}_b(x)]^{\frac{1}{p}} + [\tilde{\psi}_d(x)]^{\frac{1}{p}}\right\} \tag{6.24}$$

for all $x \in X$, *where* $\tilde{\psi}_b(x)$ *and* $\tilde{\psi}_d(x)$ *for all* $x \in X$ *are defined as in Theorems 6.11 and 6.14 respectively.*

Corollary 6.19. *Let* $\nu \ge 0$ *and* r, s *and* t *which are all* > 0 *be real numbers such that* r, s *and* t *are all* $\neq 2$ *and* 4. *Suppose that a mapping* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality* (3.23) *for all* $x, y, z \in X$. *Then there exist a unique quadratic mapping* $B : X \to Y$ *and a unique quartic mapping* $D : X \to Y$ *such that they satisfy the inequality* (5.8) *for all* $x \in X$, *where* α_b, α_d, $\beta_b(x)$, $\beta_d(x)$, $\gamma_b(x)$, $\gamma_d(x)$, $\delta_b(x)$, $\delta_d(x)$, $\zeta_b(x)$ *and* $\zeta_d(x)$ *are defined as in Corollaries 6.12 and 6.15*

Corollary 6.20. *Let* $\nu \ge 0$ *and* r, s *and* t *which are all* > 0 *be real numbers such that* $\lambda = r+s+t \neq 2$ *and* 4. *Suppose that a mapping* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality* (3.25) *for all* $x, y, z \in X$. *Then there exist a unique quadratic mapping* $B : X \to Y$ *and a unique quartic mapping* $D : X \to Y$ *such that they satisfy the inequality* (5.9) *for all* $x \in X$, *where* $\rho_b(x), \rho_d(x), \tau_b(x), \tau_d(x)$ *are defined as in Corollaries 6.13 and 6.16*

7 Conclusion

In this paper, we proved the Hyers-Ulam stability of the quadratic-quartic functional equation (1.6) in quasi-banach spaces.

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Competing Interests

The authors declare that no competing interests exist.

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