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# On Hamiltonian Path and Circuits in Non-Abelian Finite Groups

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***Authors' contributions***

*This work was carried out in collaboration between both authors. Author GNS designed the study, method and wrote the first draft of the manuscript. Author DS managed the literature searches, designed the program using the group application package and then pointed out some applications in real life situation. Both authors read and approved the final manuscript.*

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## ABSTRACT

The main objective of this paper is to determine the non-Abelian finite groups which contain only Abelian and Hamiltonian subgroups and to obtain some of their fundamental properties. Two exceptional groups of orders 16 and 24 were examined and are completely determined using GAP. These were achieved from the fact that if a group G contains at least one Hamiltonian subgroup and if all its subgroups are Abelian or Hamiltonian, then the group itself is Hamiltonian. We finally generate some Hamiltonian circuits in the two non-Abelian groups and then present a method of finding the number of circuits in any finite group.

*Keywords: Finite groups; non-Abelian groups; Hamiltonian path; Hamiltonian circuits.*

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## 1. INTRODUCTION

In algebra, a non-Abelian group, sometimes called a non-commutative group, is a group $(G, *)$ in which there exists at least one pair of elements $(x,\ y) \in G$ such that $x * y \neq y * x$ [1]. Most non-Abelian groups are concrete in nature and are pervasive in mathematics and physics. One of the simplest examples is the dihedral group $D_n$ and the smallest one is the one of order 6, which is isomorphic to the symmetric group $S_3$. A very common example from physics is the rotation group SO(3) in three dimensions. Rotating objects 90 degrees to the right and then 90 degrees to the left is not the same as doing them the other way round. Generally, both discrete and continuous groups may be non-Abelian. Most of the interesting Lie groups are also non-Abelian, and these play an important role in gauge theory [2].

In mathematical field of graph theory, a Hamiltonian path (also call traceable path) is a path in an undirected or directed graph that visits each vertex exactly once. A Hamiltonian path that formed a cycle is called Hamiltonian circuit (or Hamiltonian cycle) and the process of determining whether such paths and cycles exist in graphs is called Hamiltonian path problem. A graph is Hamiltonian-connected if for every pair of vertices, there is always a Hamiltonian path between the two vertices [3].

Hamiltonian paths and cycles and cycle paths are named after William Rowan Hamilton who invented the icosian game, now also known as Hamilton's puzzle, which involves finding a Hamiltonian cycle in the edge graph of the dodecahedron. Hamilton solved this problem using the icosian calculus, an algebraic structure based on roots of unity with many similarities to the quaternion (also invented by Hamilton). This solution does not generalize to arbitrary graphs.

Now, any non-Abelian Dedekind group is called a Hamiltonian group (Dedekind group is a group in which every subgroup is a normal subgroup) and in addition to some non-Abelian groups, all Abelian groups are also Dedekind groups [4]. The most familiar and smallest example of Hamiltonian group is the quaternion group of order 8, denoted by $Q_8$. Dedekind and Baer have shown (in the finite and infinite order case respectively) that every Hamiltonian group is a direct product of the form $G = Q_8 \times M \times N$, where $M$ is an elementary Abelian 2-group, and $N$ is a periodic Abelian group with all its elements of odd order. Dedekind groups are named after Richard Dedekind. He investigated them and proved some of the structure theorem for finite groups and then named the non-Abelian ones after William Rowan Hamilton, the discoverer of quaternions.

In the year 1898, George Miller delineated the structure of a Hamiltonian group in terms of its order and that of its subgroups. For instance, he shows that "a Hamilton group of order $2^m$ has $2^{2m-6}$ where $m > 3$, quaternion groups as subgroups". In 2005 Horvat *et al* used this structure to count the number of Hamiltonian groups of any order $n$, say $n = 2^e q$ where $q$ is an odd integer. If $e < 3$, there are no Hamiltonian groups of order $n$, otherwise there are the same number as there are Abelian groups of order $q$ [5].

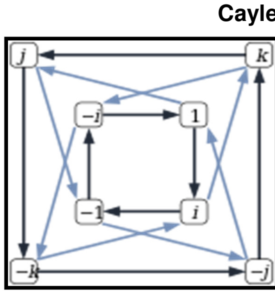
## 2. THE QUATERNION GROUP

In group theory, the quaternion group is a non-Abelian group of order 8, isomorphic to a certain eight-element subset of the quaternions under multiplication. It is often denoted by Q or $Q_8$, and is given by the group presentation

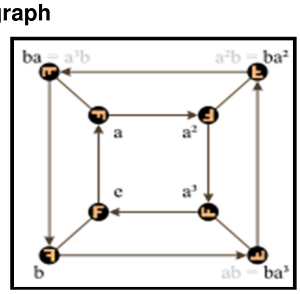
$$Q = \left\langle -1, i, j, k \mid (-1)^2 = 1, i^2 = j^2 = k^2 = ijk = -1 \right\rangle,$$

where 1 is the identity element and −1 commutes with every other elements of the group. The $Q_8$ group has the same order as the dihedral group $D_4$, but of different structure, as shown by their Cayley graphs below.

**Cayley graph**




$Q_8$




$D_4$

The black arrows in $Q_8$ represent multiplication on the right by *i*, and the blue arrows represent multiplication on the right by *j*. The dihedral group $D_4$ arises in the split quaternions in the same way that $Q_8$ lies in the quaternions. The quaternion group has the unusual property of being Hamiltonian: every subgroup of Q is a normal subgroup, but the group is non-Abelian [6] and every Hamiltonian group contains a copy of Q [7].

In abstract algebra, one can construct a real four-dimensional vector space with basis {1, *i*, *j*, *k*} and turn it into an associative algebra using multiplication as the binary operation and distributive property. This result to a skew field called the quaternions. We also note that this is not quite the same as the group algebra on Q (which would be eight-dimensional). Conversely, one can start with the quaternions and then define the quaternion group as the multiplicative subgroup consisting of eight elements {1, −1, *i*, −*i*, *j*, −*j*, *k*, −*k*}. The complex four-dimensional vector space on the same basis is called the algebra of biquaternions.

Note that all the elements *i*, *j*, and *k* have order four in Q and any two of them generate the entire group. Another presentation of Q is found in [8] as follows:

$$Q = \left\langle x, y \mid x^4 = 1, x^2 = y^2, y^{-1}xy = x^{-1} \right\rangle.$$

One may take, for instance, *i* = *x*, *j* = *y* and *k* = *x y*.

It can be seen that the center and the commutator subgroup of Q is the subgroup $Z$(Q) = {-1, +1}. The factor group Q/$Z$(Q) is isomorphic to the Klein four-group V. The inner automorphism of Q is isomorphic to Q modulo its center, and is therefore isomorphic to the Klein four-group V and the full automorphism group of Q is isomorphic to the symmetric group $S_4$ of order 24. The outer automorphism of Q is then $S_4$/V of order 6 which is isomorphic to the symmetric group $S_3$ [9].

## 3. METACYCLIC GROUP

A group *G* is said to be metacyclic if there is a cyclic normal subgroup *N* whose quotient group G/*N* is also cyclic. If G is finite, then the presentation of G contains two generators and three defining relations. Much attention has been given to some specific types of metacyclic groups by many authors such as metacyclic groups with cyclic commutator quotient as discussed by Zassenhaus and Hall [10]. And the result that every finite metacyclic group can be decomposed naturally as a semi-direct product of two Hall subgroups made by [11] was an important progress.

We shall now consider some important theorems.

**Theorem 3.1:** If a group *G* has a subgroup *N* of index 2 such that every subgroup of *N* is normal in *G*, then every Cayley diagram in *G* has a Hamiltonian path [12].

**Corollary 3.2:** If a group *G* has a cyclic subgroup of index 2, then every Cayley diagram in *G* has a Hamiltonian path. In particular every Cayley diagram in dihedral group or generalized quaternion group has a Hamiltonian path.

**Proof:** Let *N* be a cyclic subgroup of index 2 in *G*. Since [*G*: *N*] = 2, $N \triangleleft G$ [13]. Now, every subgroup of a cyclic normal subgroup is normal and so, every subgroup of *N* is normal in *G*. Thus, Theorem 3.1 applies.

We conjecture that every Cayley diagram in any metacyclic group has a Hamiltonian path. Holsztynski and Strube [14] conjectured that every Cayley diagram in a dihedral group has a Hamiltonian circuit. The following theorems show that this conjecture can be achieved by considering only generating sets which consist entirely of reflections.

**Theorem 3.3:** Let *H*U*S* generates a dihedral group, where every element of *H* is a rotation, and every element of *S* is a reflection. If Cay(*S*) has a Hamiltonian circuit, then Cay($H \cup S$) has a Hamiltonian circuit [12].

**Proof:** Let ($\beta_i$: 1 ≤ *i* ≤ *m*) be a Hamiltonian circuit in Cay(*S*) and let ($\alpha_i$ : 1 ≤ *j* ≤ *n*) be a Hamiltonian path in Cay(*H*: ($\langle H\rangle/(\langle H\rangle \cap \langle S\rangle)$)). Then {($\alpha_1$, $\alpha_2$, …, $\alpha_m$,$\beta_i$): 1 ≤ *i* ≤ *m*} is a Hamiltonian circuit in Cay(*H* ∪ *S*), and this complete the proof.

The result that Cay(*S*) has a Hamiltonian circuit whenever *S* is a single reflection or a pair of reflections yields the following corollary:

**Corollary 3.4:** If *S* generates a dihedral group and contains no more than two reflections, then Cay(*S*) has a Hamiltonian circuit.

**Proposition 3.5:** If $n$ is divisible by at most 3 distinct primes, where $n$ is any positive integer, then every Cayley diagram in the dihedral group of order 2$n$ has a Hamiltonian circuit [12].

**Proof:** Let $D_n = \langle \alpha,\beta | \alpha^n = \beta^2 = e, \beta\alpha\beta = \alpha^{-1}\rangle$. Then by the assumption on $n$, we may further assume that $S$ is a minimal generating set of the form $S = \{\beta, \beta\alpha^u, \beta\alpha^v\}$**.** Let $p$ be the largest prime divisor of $n$. Because $S$ is minimal, $p$ is a divisor of $u$, $v$ or $u - v$. Since at least one of any $p + 1$ consecutive integers is relatively prime to $n$, it follows therefore that Cay($\alpha^u$, $\alpha^v$ :$\langle\alpha\rangle$) has a Hamiltonian circuit ($\alpha_i$: $1 \le i \le n$). Hence (($\beta$, $\beta\alpha^i$): $1 \le i \le n$) is a Hamiltonian circuit in Cay($S$).

## 4. GROUPS OF ORDER 16 AND 24

We shall now consider two non-Abelian groups which contain only Abelian subgroups with their fundamental properties. This paper is devoted to a determination and a study of the non-Abelian groups which involve only Abelian and Hamiltonian subgroups. It is found that there are only two exceptions to the following theorem: If a group $G$ contains at least one Hamiltonian subgroup and if all its subgroups are either Abelian or Hamiltonian, then the group itself is Hamiltonian. The two exceptional groups are of orders 16 and 24 respectively, and are completely determined in this paper.

Now, let $G$ represent any group in which every subgroup is either Abelian or Hamiltonian and suppose that it is represented as a transitive substitution group of the lowest possible degree. If it is imprimitive, then its systems of imprimitivity are permuted according to some primitive group. In order to show that $G$ is solvable, it is only necessary to prove that this primitive group is composite since it may be simply isomorphic with $G$ and since every subgroup of $G$ is solvable. The primitive group in question is of class $n$-1, $n$ being its degree, since two of its maximal subgroups of degree $n$-1 can have only the identity element in common. And since every group of class $n$-1 and degree $n$ is composite, $G$ is solvable.

In the preceding paragraph it is not assumed that $G$ necessarily contains a Hamiltonian subgroup. The assumption may, however, be made since the non-Abelian groups in which every subgroup is Abelian are mentioned above. Hence we shall assume in what follows that $G$ contains at least one Hamiltonian subgroup. Its order can therefore not be the power of an odd prime number. We shall now consider all the possible $G^{\prime s}$ under two headings as their orders are a power of 2 or involve more than one prime factor, also examined by Miller, [15].

### 4.1 The Order |*G*| of G is $2^m$

Now, since the group $G$ involves at least one Hamiltonian subgroup, it must involve a Hamiltonian subgroup ($M$) which is composed of half of its operators. And since $M$ is Hamiltonian, three-fourths of its operators are non-invariant, of order four, and have a common square while the remaining one-fourth are the invariant operators of $M$. These invariant operators constitute an Abelian subgroup ($K$) of type (1, 1, 1, …). Moreover, with respect to the subgroup $K$, $M$ is isomorphic to the four-group and hence, there is at least one subgroup ($M'$) in $M$ which involves $K$ and one-third of the operators of order four contained in $M$, invariant under $G$. The subgroup $M'$ is thus, Abelian, of type (2, 1, 1, …). Again, since $M$ does not contain any operator whose order is greater than 4, there can be no operator in $G$ whose order exceeds 8. Moreover, every operator of order 2 is invariant under $G$. Hence, any operator of $G$ together with $K$ generates a subgroup which is either Abelian or Hamiltonian since this operator and $K$ do not generate $G$. Also, since every operator of order 2 in a Hamiltonian group is invariant, it shows that every operator of $K$ is invariant under $G$.

Supposed any other operator of order 2 were not commutative with some operator of order 4 in $M$, then the latter operator could not be in $M'$ since a Hamiltonian group involves no non-invariant operator of order 2. This shows that each operator of $M'$ is commutative with every operator of order 2 contained in $G$. But if an operator of order 2 were not commutative with any other operator of order 4 in $M$, it would then transform this operator into the product of itself with an operator of order 2 contained in $M$. Hence, the order of $G$ could not exceed 16.

We shall now show that there is one and only one group $G$ which involves operators of order 8. As an operator ($\alpha$) of order 8 in $G$ is commutative with each operator in $K$ and has its square in $M'$, it must be commutative with all operators of $M'$. But it cannot be commutative with every operator of $M$ since $\alpha^2$ transform some of these operators into their inverses. Also, as $\alpha$ transforms an operator of order 4 in $M$ into itself multiplied by an operator of order 4 which is commutative with $\alpha$, the group generated by $\alpha$ and any other operator ($\tau$) of order 4 in *M* which is not commutative with $\alpha$ is of order 16 or 32. This group must therefore coincide with *G* since it is neither Abelian nor Hamiltonian.

We proceed to show that the order of the group generated by $\sigma$ and $\alpha$ cannot be 32. If this order were 32, then $\alpha^2$ would not be in the group of order 8 generated by ($\alpha$) and the commutative subgroup of $\{\tau,\ \alpha\}$. This group and $\alpha^2$ would therefore generate the Hamiltonian group of order 16 and *G* would involve an Abelian group of type (3, 1). This Abelian subgroup would involve all the operators of order 8 contained in *G* and two of the remaining operators of order 4 would generate a non-Abelian group of order 16 involving a commutator subgroup of order 4. But such a subgroup could not occur in *G*. Thus the order of *G* cannot exceed 16. Now, as the groups of order 16 are known, we may state the result as follows: There is one and only one group of order $2^m$, $m \in \Re^+$, which involves operators whose orders exceed four and satisfies the additional conditions that every subgroup is either Abelian or Hamiltonian, and that at least one subgroup is Hamiltonian. This is the group of order 16 which contains a cyclic subgroup of order 8, where the remaining operators are of order 4 and transform each operator of this cyclic subgroup into their inverses.

Moreover, half of the operators of *G* which are not in *M* transform the operators of order 4 in *M* into their inverses while the remaining half are commutative with these operators. Hence, *M* is contained in two Hamiltonian and one Abelian subgroup of order $\frac{1}{2}|G|$. We now proceed to prove that *G* contains operators of order 2 which are not in *M* whenever its order exceeds 16. Let *t* represent an operator of *G* not in *M* and which transforms the operators of order 4 in *M* into their inverses. The operators of order 4 in *M* and *t* would have a common square and the group generated by *t* and an operator of *M* which is not contained in *M* is at most, of order 16. But it could not be of this order since any two operators of a Hamiltonian group cannot generate a group whose order is 16. Therefore, it follows that *t* is commutative with half of the operators of *M* and hence with operators of order 4 in *M*. And as the product of *t* with such an operator is of order 2, *G* is the direct product of *M* and an operator of order 2. Thus, if every subgroup of a group of order $2^m$, $m > 4$, is either Abelian or Hamiltonian and if there is at least one Hamiltonian subgroup, then the entire group is Hamiltonian.

Finally, we shall now consider groups of order 16. If such a group of order 16 contains a Hamiltonian subgroup, it must be the quaternion group. As every group of order 16 contains an Abelian subgroup of order 8 and as the groups under consideration do not involve any operators of order 8, it then follows that *G* would have to contain operators of order 2 in addition to the one contained in the quaternion subgroup. Also such an operator has to be commutative with each operator in the quaternion subgroup. Thus *G* itself is the Hamiltonian group of order 16, and it shows that the group of order 16 considered above is the only non-Hamiltonian group of order $2^m$ in which every subgroup is either Abelian or Hamiltonian and in which at least one subgroup is Hamiltonian. Hence with this single exception, the Hamiltonian groups are the only ones involving Hamiltonian subgroups, with no other non-Abelian subgroups.

### 4.2 |*G*| is Divisible by at Least Two Distinct Prime Numbers

Since *G* contained at least one Hamiltonian subgroup, its order is divisible by 8. Therefore it contains an invariant subgroup *M* of prime index (*p*) since it is solvable. But this subgroup is the direct product of its own Sylow *p*-subgroups since it is either Hamiltonian or Abelian, and the order of every operator of *G* which is not also in *M* is divisible by *p*. Thus if *t* represent one of these operators, then $t^p$ is in *M*. Also each of the Sylow subgroups of *M* is transformed into itself by *t*. Now when $p = 2$, then *M* involves at least two different Sylow subgroups and *t* is commutative with all the operators of odd order contained in *M*. Thus, the group generated by *t* and the Sylow *p*-subgroup of order $2^m$ contained in *M* is Hamiltonian since it is a Sylow subgroup of *G*. Hence, *G* is the direct product of its Sylow subgroups whenever it contains an invariant subgroup of half its order as all of its operators are commutative with every operator of odd order in *M*. Also, the Sylow subgroup of order $2^{m+1}$ is Hamiltonian while all the others are Abelian. Moreover, any such direct product is always a group of the required kind. Assuming that *p* is odd. When $p > 2$, the Sylow subgroup of order $2^m$ contained in *M* is obviously Hamiltonian and when *M* involves more than one Sylow subgroup, then *t* is commutative with every operator of *M* since it is commutative with all the operators in its Sylow subgroups. Hence *G* is the direct product of its Sylow subgroups whenever the order of $M$ is divisible by at least two different prime numbers. It remains now to consider the case when $M$ is the Hamiltonian group of order $2^m$ and $p > 2$.

Since $M$ contains only three Abelian subgroups of order $2^{m-1}$, $t$ would have to transform each of these into itself whenever $p > 3$. As the group generated by $t$ and such a subgroup would be Abelian, $t$ would again be commutative with every operator of $M$ and we have proved that if the order of $G$ is not of the form $2^m.3$, then $G$ is always the direct product of an Abelian group of odd order and a Hamiltonian group of order $2^m$. We shall now assume that the order of $G$ is the product $2^m.3$ and then consider the possible value of $m$. But the order of the subgroup generated by three operators of order 4 in $M$ cannot exceed 16. Then assuming that $t$ is not commutative with every operator of order 4 in $M$. It would therefore transform a subgroup of $M$ whose order cannot exceed 16 into itself. Since the group generated by $t$ and this subgroup is neither Abelian nor Hamiltonian, the order of $G$ cannot exceed 48 unless it is the direct product of its Sylow subgroups. Therefore we may now assume that $t$ does not transform each of the three Abelian subgroups of order $2^{m-1}$ into itself, otherwise it would be commutative with each operator of $M$, and hence the group which the three conjugate operators of order 4 generate is the Hamiltonian group of order 16 or the quaternion group. Now, as the group contains exactly four quaternion subgroups, $t$ and this quaternion subgroup generate the group of order 24 which has no subgroup of order 12. Hence we have the following conclusion:

> If a group $G$ contains at least one Hamiltonian subgroup and if all its subgroups are either Abelian or Hamiltonian, then it is the direct product of the Hamiltonian group of order $2^m$ for some positive integer $m$ and an Abelian group of odd order, unless it is the group of order 24 which does not contain a subgroup of order 12. Hence, there are only two non-Hamiltonian groups which contain at least one Hamiltonian subgroup and whose other subgroups are either Abelian or Hamiltonian. The orders of these groups are 16 and 24 respectively, while there are infinite number of non-Abelian groups in which every subgroup is Abelian.

## 5. THE GROUPS $D_8$ AND $Q_{12}$

We shall start this section by considering the group $D_8$ of order 16 as follows:

```
gap> D:=DihedralGroup(IsPermGroup, 16);
Group([ (1,2,3,4,5,6,7,8), (2,8)(3,7)(4,6) ])
gap> r:= D.1;
(1,2,3,4,5,6,7,8)
gap> s:= D.2;
(2,8)(3,7)(4,6)
gap> M:= Subgroup(D, [r]);
Group([ (1,2,3,4,5,6,7,8) ])
gap> N:= Subgroup(D, [s]);
Group([ (2,8)(3,7)(4,6) ])
gap> Centre(D);
Group([ (1,5)(2,6)(3,7)(4,8) ])
gap> IsCyclic(M);
true
gap> IsCyclic(N);
true
gap> IsCyclic(Center(D));
true
gap> Size(M); Size(N); Size(Centre(D));
8
2
2

gap> Read("CircuitCheck");
gap> CircuitCheck((1,2,3,4,5,6,7,8));
[ (1,2,3,4,5,6,7,8), (1,3,5,7)(2,4,6,8,9,10,11,12), (1,4,7,9,11,2,5,8,10,12,3,6), (1,5)(2,6,9,12,4,8,11,3,7,10),
  (1,6,10,3,8,12,5,9,2,7,11,4), (1,7,12,6,11,5,10,4,9,3)(2,8), (1,8,3,9,4,10,5,11,6,12,7,2),
  (2,9,5,12,8,4,11,7,3,10,6), (1,9,6,3,11,8,5,2,10,7,4,12), (1,10,8,6,4,2,11)(3,12,9,7,5),
  (1,11,9,8,7,6,5,4,3,2,12,10), (1,12,11,10,9), (1,10,2,9,3)(4,8)(5,7)(11,12), (1,9,2)(3,8)(4,7)(5,6)(10,12),
```

```
  (2,8)(3,7)(4,6)(9,12)(10,11), (1,8,12)(2,7)(3,6)(4,5)(9,11), (1,7,12,8,11)(2,6)(3,5)(9,10),
  (1,6,12,7,11,8,10)(2,5)(3,4), (1,5,12,6,11,7,10,8,9)(2,4), (1,4,12,5,11,6,10,7,9,8)(2,3), (1,3,12,4,11,5,10,6,9,7),
  (1,2,12,3,11,4,10,5,9,6)(7,8), (1,12,2,11,3,10,4,9,5)(6,8), (1,11,2,10,3,9,4)(5,8)(6,7), (1,2,3,4,5,6,7,8) ]
```

```
gap> CircuitCheck((1,3)(2,4)(5,7)(6,8));
[ (1,3)(2,4)(5,7)(6,8), (1,4,7,6,5,8,9,10,11,12,3,2), (3,7,9,11)(4,8,10,12), (1,2,7,10,3,8,11,4,5,6,9,12),
  (1,7,11)(2,8,12)(3,5,9)(4,6,10), (1,8,3,6,11,2,5,10)(4,9)(7,12), (1,5,11,7,3,9)(2,6,12,8,4,10),
  (1,6,3,10,7,4,11,8)(2,9)(5,12), (1,9,7)(2,10,8)(3,11,5)(4,12,6), (1,10,5,4,3,12,9,8,7,2,11,6),
  (1,11,9,5)(2,12,10,6), (1,12,11,10,9,6,7,8,5,2,3,4), (1,10,3,6,7,2,9,4,5,8)(11,12), (1,9,3,5,7)(2,6)(4,8)(10,12),
  (1,6)(2,5)(3,8)(4,7)(9,12)(10,11), (1,5)(2,8,12,6,4)(3,7)(9,11), (1,8,11,6,3,2,7,12,5,4)(9,10),
  (1,7,11,5,3)(6,12,8,10), (1,2)(3,4)(5,12,7,10)(6,11,8,9), (2,4,12)(5,11,7,9)(6,10,8), (1,4,11,2,3,12)(5,10,7,6,9,8),
  (1,3,11)(2,12,4,10)(5,9,7), (1,12,3,10)(2,11,4,9)(5,6)(7,8), (1,11,3,9)(2,10,4,6,8), (1,3)(2,4)(5,7)(6,8) ]
```

```
gap> CircuitCheck((1,7)(2,6)(3,5));
[ (1,7)(2,6)(3,5), (1,6)(2,5)(3,4)(7,8,9,10,11,12), (1,5)(2,4)(6,8,10,12)(7,9,11), (1,4)(2,3)(5,8,11,6,9,12)(7,10),
  (1,3)(4,8,12)(5,9,7,11)(6,10), (1,2)(3,8,7,12)(4,9,6,11)(5,10), (2,8,6,12)(3,9,5,11)(4,10),
  (1,8,5,12)(2,9,4,11)(3,10)(6,7), (1,9,3,11)(2,10)(4,12,8)(5,7), (1,10)(2,11,8,3,12,9)(4,7)(5,6),
  (1,11,9)(2,12,10,8)(3,7)(4,6), (1,12,11,10,9,8)(2,7)(3,6)(4,5), (1,10,7,4)(2,9,6,3,8,5)(11,12),
  (1,9,7,5,3)(2,8,6,4)(10,12), (1,8,7,6,5,4,3,2)(9,12)(10,11), (8,12)(9,11), (1,2,3,4,5,6,7,12)(8,11)(9,10),
  (1,3,5,7,11)(2,4,6,12)(8,10), (1,4,7,10)(2,5,12,3,6,11)(8,9), (1,5,11,3,7,9)(2,6,10)(4,12),
  (1,6,9,2,7,8)(3,12,5,10)(4,11), (1,7)(2,12,6,8)(3,11,5,9)(4,10), (1,12,7,2,11,6)(3,10,5,8)(4,9),
  (1,11,7,3,9,5)(2,10,6)(4,8), (1,7)(2,6)(3,5) ]
```

Next, we consider the *double dihedral group* $Q_{12}$ of order 24. Algebraically, the *double dihedral group* $Q_{12}$ is the group generated by two elements $\alpha$ (rotation) and $\beta$ (reflection) subject to the following relations; $\alpha^{12} = e$, $\beta^2 = e$, $\alpha\beta\alpha = \beta$, and $\beta\alpha\beta = \alpha^{-1}$. Hence, by the appropriate program in GAP, we have the following result:

```
gap> G:= DihedralGroup(IsPermGroup, 24);
Group([ (1,2,3,4,5,6,7,8,9,10,11,12), (2,12)(3,11)(4,10)(5,9)(6,8) ])
gap> a:= G.1;
(1,2,3,4,5,6,7,8,9,10,11,12)
gap> H:= Subgroup(G, [a]);
Group([ (1,2,3,4,5,6,7,8,9,10,11,12) ])
gap> b:= G.2;
(2,12)(3,11)(4,10)(5,9)(6,8)
gap> J:= Subgroup(G, [b]);
Group([ (2,12)(3,11)(4,10)(5,9)(6,8) ])
gap> K:= Subgroup(G, [a,b]);
Group([ (1,2,3,4,5,6,7,8,9,10,11,12), (2,12)(3,11)(4,10)(5,9)(6,8) ])
gap> Center(G);
Group([ (1,7)(2,8)(3,9)(4,10)(5,11)(6,12) ])
gap> Elements(Center(G));
[ (), (1,7)(2,8)(3,9)(4,10)(5,11)(6,12) ]
gap> IsAbelian(G);
false
gap> IsAbelian(H);
true
gap> IsAbelian(J);
true
gap> IsCyclic(G);
false
gap> IsCyclic(H);
true
gap> IsCyclic(J);
true
gap> Size(G); Size(H); Size(J);
24
12
2
gap> Size(Center(G));
2
gap> Center(G) = J;
false
```

From the above results, it is obvious that both $D_8$ and $Q_{12}$ are not Abelian but the subgroup generated by the rotations and the reflections from each group is Abelian. The order of the centre of each group is 2 and in both cases, each reflection generates a cyclic subgroup J of order 2 since $o(\beta) = 2$ for each reflection $\beta$. These subgroups are normal. Again, if H is the subgroup consisting of all the rotations in any of the groups, then H is obviously a normal subgroup since [G:H] = 2, [16]. Hence by induction, all subgroups of the groups $D_8$ and $Q_{12}$ are Abelian and cyclic.

We shall now write the appropriate program in GAP to obtain some Hamiltonian circuits in the group $Q_{12}$ as follows:

```
gap> Read("CircuitCheck");
gap> CircuitCheck(());
[ (), (1,2,3,4,5,6,7,8,9,10,11,12), (1,3,5,7,9,11)(2,4,6,8,10,12), (1,4,7,10)(2,5,8,11)(3,6,9,12),
  (1,5,9)(2,6,10)(3,7,11)(4,8,12), (1,6,11,4,9,2,7,12,5,10,3,8), (1,7)(2,8)(3,9)(4,10)(5,11)(6,12),
  (1,8,3,10,5,12,7,2,9,4,11,6), (1,9,5)(2,10,6)(3,11,7)(4,12,8), (1,10,7,4)(2,11,8,5)(3,12,9,6),
  (1,11,9,7,5,3)(2,12,10,8,6,4), (1,12,11,10,9,8,7,6,5,4,3,2), (1,10)(2,9)(3,8)(4,7)(5,6)(11,12),
  (1,9)(2,8)(3,7)(4,6)(10,12), (1,8)(2,7)(3,6)(4,5)(9,12)(10,11), (1,7)(2,6)(3,5)(8,12)(9,11),
  (1,6)(2,5)(3,4)(7,12)(8,11)(9,10), (1,5)(2,4)(6,12)(7,11)(8,10), (1,4)(2,3)(5,12)(6,11)(7,10)(8,9),
  (1,3)(4,12)(5,11)(6,10)(7,9), (1,2)(3,12)(4,11)(5,10)(6,9)(7,8), (2,12)(3,11)(4,10)(5,9)(6,8),
  (1,12)(2,11)(3,10)(4,9)(5,8)(6,7), (1,11)(2,10)(3,9)(4,8)(5,7), () ]

gap> CircuitCheck((1,2,3,4,5,6,7,8,9,10,11,12));
[ (1,2,3,4,5,6,7,8,9,10,11,12), (1,3,5,7,9,11)(2,4,6,8,10,12), (1,4,7,10)(2,5,8,11)(3,6,9,12),
  (1,5,9)(2,6,10)(3,7,11)(4,8,12), (1,6,11,4,9,2,7,12,5,10,3,8), (1,7)(2,8)(3,9)(4,10)(5,11)(6,12),
  (1,8,3,10,5,12,7,2,9,4,11,6), (1,9,5)(2,10,6)(3,11,7)(4,12,8), (1,10,7,4)(2,11,8,5)(3,12,9,6),
  (1,11,9,7,5,3)(2,12,10,8,6,4), (1,12,11,10,9,8,7,6,5,4,3,2), (), (1,11)(2,10)(3,9)(4,8)(5,7),
  (1,10)(2,9)(3,8)(4,7)(5,6)(11,12), (1,9)(2,8)(3,7)(4,6)(10,12), (1,8)(2,7)(3,6)(4,5)(9,12)(10,11),
  (1,7)(2,6)(3,5)(8,12)(9,11), (1,6)(2,5)(3,4)(7,12)(8,11)(9,10), (1,5)(2,4)(6,12)(7,11)(8,10),
  (1,4)(2,3)(5,12)(6,11)(7,10)(8,9), (1,3)(4,12)(5,11)(6,10)(7,9), (1,2)(3,12)(4,11)(5,10)(6,9)(7,8),
  (2,12)(3,11)(4,10)(5,9)(6,8), (1,12)(2,11)(3,10)(4,9)(5,8)(6,7), (1,2,3,4,5,6,7,8,9,10,11,12) ]

gap> CircuitCheck((2,12)(3,11)(4,10)(5,9)(6,8));
[ (2,12)(3,11)(4,10)(5,9)(6,8), (1,12)(2,11)(3,10)(4,9)(5,8)(6,7), (1,11)(2,10)(3,9)(4,8)(5,7),
  (1,10)(2,9)(3,8)(4,7)(5,6)(11,12), (1,9)(2,8)(3,7)(4,6)(10,12), (1,8)(2,7)(3,6)(4,5)(9,12)(10,11),
  (1,7)(2,6)(3,5)(8,12)(9,11), (1,6)(2,5)(3,4)(7,12)(8,11)(9,10), (1,5)(2,4)(6,12)(7,11)(8,10),
  (1,4)(2,3)(5,12)(6,11)(7,10)(8,9), (1,3)(4,12)(5,11)(6,10)(7,9), (1,2)(3,12)(4,11)(5,10)(6,9)(7,8),
  (1,4,7,10)(2,5,8,11)(3,6,9,12), (1,5,9)(2,6,10)(3,7,11)(4,8,12), (1,6,11,4,9,2,7,12,5,10,3,8),
  (1,7)(2,8)(3,9)(4,10)(5,11)(6,12), (1,8,3,10,5,12,7,2,9,4,11,6), (1,9,5)(2,10,6)(3,11,7)(4,12,8),
  (1,10,7,4)(2,11,8,5)(3,12,9,6), (1,11,9,7,5,3)(2,12,10,8,6,4), (1,12,11,10,9,8,7,6,5,4,3,2), (),
  (1,2,3,4,5,6,7,8,9,10,11,12), (1,3,5,7,9,11)(2,4,6,8,10,12), (2,12)(3,11)(4,10)(5,9)(6,8) ]

gap> CircuitCheck((1,3,5,7,9,11)(2,4,6,8,10,12));
[ (1,3,5,7,9,11)(2,4,6,8,10,12), (1,4,7,10)(2,5,8,11)(3,6,9,12), (1,5,9)(2,6,10)(3,7,11)(4,8,12),
  (1,6,11,4,9,2,7,12,5,10,3,8), (1,7)(2,8)(3,9)(4,10)(5,11)(6,12), (1,8,3,10,5,12,7,2,9,4,11,6),
  (1,9,5)(2,10,6)(3,11,7)(4,12,8), (1,10,7,4)(2,11,8,5)(3,12,9,6), (1,11,9,7,5,3)(2,12,10,8,6,4),
  (1,12,11,10,9,8,7,6,5,4,3,2), (), (1,2,3,4,5,6,7,8,9,10,11,12), (1,12)(2,11)(3,10)(4,9)(5,8)(6,7),
  (1,11)(2,10)(3,9)(4,8)(5,7), (1,10)(2,9)(3,8)(4,7)(5,6)(11,12), (1,9)(2,8)(3,7)(4,6)(10,12),
  (1,8)(2,7)(3,6)(4,5)(9,12)(10,11), (1,7)(2,6)(3,5)(8,12)(9,11), (1,6)(2,5)(3,4)(7,12)(8,11)(9,10),
  (1,5)(2,4)(6,12)(7,11)(8,10), (1,4)(2,3)(5,12)(6,11)(7,10)(8,9), (1,3)(4,12)(5,11)(6,10)(7,9),
  (1,2)(3,12)(4,11)(5,10)(6,9)(7,8), (2,12)(3,11)(4,10)(5,9)(6,8), (1,3,5,7,9,11)(2,4,6,8,10,12) ]

gap> CircuitCheck((1,10,7,4)(2,11,8,5)(3,12,9,6));
[ (1,10,7,4)(2,11,8,5)(3,12,9,6), (1,11,9,7,5,3)(2,12,10,8,6,4), (1,12,11,10,9,8,7,6,5,4,3,2), (),
  (1,2,3,4,5,6,7,8,9,10,11,12), (1,3,5,7,9,11)(2,4,6,8,10,12), (1,4,7,10)(2,5,8,11)(3,6,9,12),
  (1,5,9)(2,6,10)(3,7,11)(4,8,12), (1,6,11,4,9,2,7,12,5,10,3,8), (1,7)(2,8)(3,9)(4,10)(5,11)(6,12),
  (1,8,3,10,5,12,7,2,9,4,11,6), (1,9,5)(2,10,6)(3,11,7)(4,12,8), (1,7)(2,6)(3,5)(8,12)(9,11),
  (1,6)(2,5)(3,4)(7,12)(8,11)(9,10), (1,5)(2,4)(6,12)(7,11)(8,10), (1,4)(2,3)(5,12)(6,11)(7,10)(8,9),
  (1,3)(4,12)(5,11)(6,10)(7,9), (1,2)(3,12)(4,11)(5,10)(6,9)(7,8), (2,12)(3,11)(4,10)(5,9)(6,8),
  (1,12)(2,11)(3,10)(4,9)(5,8)(6,7), (1,11)(2,10)(3,9)(4,8)(5,7), (1,10)(2,9)(3,8)(4,7)(5,6)(11,12),
  (1,9)(2,8)(3,7)(4,6)(10,12), (1,8)(2,7)(3,6)(4,5)(9,12)(10,11), (1,10,7,4)(2,11,8,5)(3,12,9,6) ]
```

Sometimes one may ask question like "Does any non-trivial finite group have a Hamilton circuit?" The obvious answer is always *yes*. But the next question may turn out to be "how many different Hamilton circuits does it have?" Now, to answer this question, we shall consider a set of four different objects and denote it by $V_4$, so that $V_4$ = {A, B, C, D}. Now, if we move through the four points of $V_4$ in an arbitrary order, we get a Hamilton path. Take for example, C→B→D→A (= B→D→A→C = D→A→C→B = A→C→B→D) is a Hamilton path; D→C→B→A (= C→B→A→D = B→A→D→C = A→D→C→B) is another Hamiltonian path; and so on. Each of these Hamilton paths can be closed into a Hamilton circuit. The path C→B→D→A begets the circuit C→B→D→A→C; the path D→C→B→A begets the circuit D→C→B→A→D; and so on. Hence, in each non-trivial finite group, there is abundance of Hamilton circuits.

Now, to find all the Hamiltonian circuits in $V_4$, for simplicity, we shall consider each circuit just once, using a common reference point say A because as long as we are consistent, it doesn't really matter which reference point we shall pick. Therefore each Hamilton circuits will be described by a sequence that starts and ends with the object A, with the objects B, C and D, permuted in between in some order. Thus, there are a total of $3 \times 2 \times 1 = 6$ different ways to shuffle the objects B, C, and D, each producing a different Hamilton circuit in $V_4$. Therefore by induction, there are $(n-1) \times (n-2) \times \ldots \times 2 \times 1 = (n-1)!$ Hamilton circuits in any set $V_n$ consisting of $n$ objects.

We shall now find the number of Hamiltonian circuits in $D_8$ and $Q_{12}$ as follows:

```
gap> Factorial(15);
1307674368000

gap> Factorial(23);
25852016738884976640000
```

Hence, there are 1307674368000 and 25852016738884976640000 Hamiltonian circuits in $D_8$ and $Q_{12}$ respectively.

## 6. CONCLUSION

We have successfully shown that all subgroups of the non-Abelian groups $D_8$ and $Q_{12}$ are Abelian and cyclic where $Z(D_8)$ and $Z(Q_{12})$ are found to be doublets (i.e. each of order 2). These subgroups are Hamiltonian. Hence, if a group $G$ contains at least one Hamiltonian subgroup and if all its subgroups are either Abelian or Hamiltonian, then it can be express as the direct product of the Hamiltonian group of order $2^m$ for some positive integer $m$ and an Abelian group of odd order, unless it is the group of order 24 which does not contain a subgroup of order 12. It was also shown that every finite non-trivial group contains a Hamiltonian circuit and we were able to generate some Hamiltonian circuits in the non-Abelian finite groups $D_8$ and $Q_{12}$, and the total number of the circuits in each group was determined using GAP.

This concept of Hamiltonian circuits in finite groups is very useful in electrical network and even computer networking which is to be addressed in our next paper. Nevertheless, we shall give a concrete example in our daily routine. Consider a school bus for a certain school. We call such school bus "the traveler" which always lives school in the morning time and picks up children at the designated bus stops which is called "the sites" and drops them off at the end of the day at the same sites. For a typical school bus route, there may be 10 to 20 such sites. In this case, total time on the bus is always an important variable because students have to get to school on time, and there is a known time of travel between any two sites. Since children must be picked up at every site, a tour of all the sites, starting and ending at the school is required. And since the bus repeats its route every school day throughout the session, finding an optimal tour is crucial, and each route, starting and ending at the school, is called a Hamiltonian circuit.

## COMPETING INTERESTS

Authors have declared that no competing interests exist.

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