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Application of Scalar Type Operators to Decomposability

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

In this paper, we give some application of scalar type operators to Decomposibility. In particular, we show that if H is of $(\alpha, \alpha+1)$type $\mathbf{R}$ and that it generates a strongly continuous group on a Banach space, then its resolvent is Decomposable hence scalar type.

Keywords: Decomposable operators; Bishop's property.

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1 Introduction

Definition: Decomposable Operator A bounded operator H on a complex Banach space X is decomposable provided that whenever $\{U_1, U_2, ..., U_n\}$ is an open cover of $\mathbf{C}$, there exists closed, H-invariant subspaces Y_k such that $X = Y_1 + Y_2 + ... + Y_n$ and $\sigma(H \mid Y_k) \subseteq U_k$, $k = 1, 2, ..., n$.

This class of operators contains all normal operators on a Hilbert space and compact Banach space operators hence they are of $(\alpha, \alpha+1)$ type $\mathbf{R}$ operators [1]. The following Theorem due to Albrecht

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and Eschmier [2] gives the necessary and sufficient condition for a bounded operator $H \in B(X)$ to be decomposable.

Theorem 1 [2]: A bounded operator $H \in B(X)$ is decomposable if and only if H has Bishop's property (β) and the decomposition property (δ).

Definition: Let X be a Banach space and Ω an open subset of the plane. Let $Hol(\Omega, X)$ denote the space of analytic functions from Ω to X. Then $Hol(\Omega, X)$ is a Fretchet space with respect to uniform convergence on the compact subsets of Ω. The operator $H \in B(X)$ is said to possess Bishop's property (β), provided that for every open subset $\Omega \subset \mathbf{C}$, $H_\Omega : Hol(\Omega, X) \to Hol(\Omega, X)$, $H_\Omega f(z) = (z - H)f(z)$ is injective with closed range.

[Decomposition Property (δ)]

If F is a closed subspace of $\mathbf{C}$, then the glocal analytic spectral subspace $X_H(F)$ is $X_H(F) = X \cap ranH_{\mathbf{C}\setminus F}$, that is $x \in X_H(F)$ if there exist an analytic function $f : \mathbf{C} \setminus F \to X$ so that $(H - \lambda)f(\lambda) = x$ for all $\lambda \in \mathbf{C} \setminus F$ A bounded linear operator $H \in B(X)$ has the decomposition property (δ) if $X = X_H(\overline{U}) + X_H(\overline{V})$ for every open cover $\{U, V\}$ of $\mathbf{C}$.

Albrecht and Escheneier [2] established the remarkable fact that the properties (β) and (δ) are dual to each other. Indeed, $H \in B(X)$ has property (β) (resp (δ)) if and only if H^* has (δ) (resp.(β)).

We shall greatly use the following formulation by Laursen and Neumann [3]

Theorem 2 [3]: Let $H \in B(X)$ and D be a closed disk that contains $\sigma(H)$, and let V be an open neighborhood of D. Suppose that there exist a totally disconnected compact subset E of the boundary of D, a locally bounded function $\omega : V \setminus E \to (0, \infty)$ and an increasing function $\gamma : (0, \infty) \to (0, \infty)$ such that log of γ has an integrable singularity at zero and $\gamma(dist(\lambda, \partial D)) \parallel x \parallel \leq \omega(\lambda) \parallel (H - \lambda)x \parallel$ for all $x \in X$ and $\lambda \in V \setminus \partial D$, then H has property (β)

In particular, the above Theorem provide sufficient conditions in terms of the norms of resolvents sufficient for bishop's property (β).

Lemma 3: Let H be generator of arbitrarily continuous semigroup on a Banach space X and let $\lambda, \mu \in \rho(H)$, then $R(\lambda, H)R(\mu, H) = R(\mu, H)R(\lambda, H)$

Proof: The proof follows immediately from the well known resolvent identity;

$R(\lambda, H) - R(\mu, H) = -(\mu - \lambda)R(\lambda, H)R(\mu, H)$ for all $\lambda, \mu \in \rho(H)$.

Lemma 4: Let H be as in Lemma [3] and let $T = R(\lambda, H)$. Then $\mu \in \rho(T)$ if and only if $\lambda - \frac{1}{\mu} \in \rho(T)$. In this case, we have

$$(\mu - T)^{-1} = \frac{1}{\mu}I + \frac{1}{u^2}R(\lambda - \frac{1}{\mu}, H) \tag{1.1}$$

From equation (1), $R(\lambda - \frac{1}{\mu}, H) - T = [\lambda - (\lambda - \frac{1}{\mu})]TR(\lambda - \frac{1}{\mu}, H)$ which implies $\mu T = (\mu - T)R(\lambda - \frac{1}{\mu}, H) = R(\lambda - \frac{1}{\mu}, H)(\mu - T)$. Multiplying by $(\lambda - H)$ and dividing through by μ yields

$$I = \frac{1}{\mu}(\lambda - \frac{1}{\mu} - H + \frac{1}{\mu})R(\lambda - \frac{1}{\mu}, H)(\mu - T)$$

$$= \frac{1}{\mu}(I + \frac{1}{\mu}R(\lambda - \frac{1}{\mu}, H))(\mu - T)$$

Thus

$$(\mu - T)^{-1} = \frac{1}{\mu}I + \frac{1}{\mu^2}R(\lambda - \frac{1}{\mu}, H).$$

The next theorem which is a major result in this section indicates that the kind of resolvent we are dealing with here are decomposable.

Theorem 5: If H is a generator of arbitrarily strongly continuous semigroup on Banach Space X with $\sigma(H, X) \subset \{z : Re(z) \leq c\}$ on a Banach space X, then the resolvent operator $R(\lambda, H)$ is decomposable for all $\lambda \in \rho(H, X)$.

Proof: Let H be the generator of strongly continuous semigroup with $\sigma(H, X) \subset \{z : Re(z) \leq c\}$ on a Banach space X. Let $\lambda, \mu \in \rho(H)$ and $T = R(\lambda, H)$. By the Hille Yosida theorem we have

$$\| R(\lambda - \frac{1}{\mu}, H) \| \leq \frac{M}{Re(\lambda - \frac{1}{\mu}) - c}$$

where $M > 0$ is a constant.
Now, by the spectral mapping theorem, we get

$$\sigma(T) = \{\frac{1}{\lambda - it} : t \in \mathbf{R}\} \cup \{0\}$$

letting $\omega = \frac{1}{\lambda - it}$ where $\lambda = Re(\lambda) + iIm(\lambda)$

$$\sigma(T) = \{\omega : \mid \omega - \frac{1}{2Re(\lambda)} \mid = \frac{1}{2Re(\lambda)},\ Re(\lambda) > 0\}$$

For any $\mu \in \rho(T)$, we have $\mid \mu - \frac{1}{2\lambda} \mid > \frac{1}{2\lambda}$ which implies $Re(\lambda - \frac{1}{\mu}) > 0$ and thus $dist(\mu, \sigma(T)) = Re(\lambda - \frac{1}{\mu})$. Consequently,

$$\| R(\lambda - \frac{1}{\mu}, H) \| \leq \frac{M}{dist(\mu, \sigma(T))}$$

And from Lemma 4, we obtain

$$\| R(\mu, T) \| \leq \frac{1}{\mid \mu \mid} + \frac{1}{\mid \mu \mid^2} dist(\mu, \sigma(T))$$

It follows from Theorem 2 that T has Bishop property (β). Moreover, the adjoint operator T^* satisfies $\sigma(T^*) = \sigma(T)$ and thus

$$\| R(\mu, T^*) \| \leq \frac{1}{\mu} + \frac{1}{\mu^2} dist(\mu, \sigma(T))$$

which indicates that T^* has Bishop's property (β). This implies that T has property (δ). Thus by Theorem 1 it follows that H is decomposable.

2 Hardy Spaces

Let $\mathbf{D} = \{z \in \mathbf{C} : \mid z \mid < 1\}$ denote the unit disk of the complex plane and $H(\mathbf{D})$ denote the Frechet space of functions analytic on $\mathbf{D}$. For $0 < p < \infty$, the hardy spaces on the unit disk, $H^p(\mathbf{D})$ are defined as $H^p(\mathbf{D}) = \{f \in H(\mathbf{D}) : \| f \|_{H^p(\mathbf{D})} = \sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mid f(re^{i\theta}) \mid^p d\theta < \infty\}$. We refer to [4] for the basic and comprehensive theory of Hardy spaces. In particular, it is important to note that every $f \in H^p(\mathbf{D})$, $0 < p < \infty$, has non tangential boundary values almost everywhere on $\partial\mathbf{D}$ and

$$\| f \|_{H^p(\mathbf{D})} = \left(\int_0^{2\pi} \mid f(e^{i\theta}) \mid^p d\theta \right)^{\frac{1}{p}}$$

Where we regard the boundary function as an extension of f. Moreover the growth condition for the functions in $H^p(\mathbf{D})$ is given by

$$| f(z) |^p \leq \frac{1}{1- | z |^2} \| f \|^p_{H^p(D)}$$

$1 \leq p < \infty$, $f \in H^p(\mathbf{D})$.

We consider the following self analytic map $\varphi_t : \mathbf{D} \to \mathbf{D}$ given by

$$\varphi_t(z) = e^{-ct} z$$

for all $z \in \mathbf{D}$, $t > 0$. We define the corresponding weighted composition operators on $H^p(\mathbf{D})$ by

$$\begin{aligned} T_t f(z) &= (\varphi_t'(z))^\gamma f(\varphi_{t(z)}) \\ &= e^{-ct\gamma} f(e^{-ct} z) \end{aligned}$$

for all $f \in H^p(\mathbf{D})$, $\gamma = \frac{1}{p}$.

3 Main Results

The following theorem gives both the semigroup and spectral properties of this group $\{T_t\}$ of composition operators.

Theorem 6: Let $H^p(\mathbf{D}), 1 \leq p \leq \infty$ be hardy space of the unit disk $\mathbf{D}$. Define a self analytic map $\varphi_t : \mathbf{D} \to \mathbf{D}$ by $\varphi_t(z) = e^{-ct} z$ and the corresponding weighted composition operator $T_t : H^p(\mathbf{D}) \to H^p(\mathbf{D})$ by $T_t f(z) = e^{-ct\gamma} f(e^{-ct} z)$ where $c \in \mathbf{C}$, $t \geq 0$ and $\gamma = \frac{1}{p}$. Then the following hold:

(a) $(T_t)_{t \in \mathbf{R}}$ is a group of isometries on $H^p(\mathbf{D})$

(b) $(T_t)_{t \in \mathbf{R}}$ is strongly continuous.

(c) The infinitesimal generator H of T_t is given by $Hf(z) = -cHf(z) - czf'(z)$ with the domain $dom(H) = \{f \in H^p(\mathbf{D}) : zf'(z) \in H^p(\mathbf{D})\}$

(d) $\sigma(H) = \sigma_p(H) = \{-c(n + \frac{1}{p}) : n = 0, 1, 2, ...\}$

(e) If $Re(c) = 0$, then $R(c, H)$ is compact, decomposable and a scalar type operator.

Proof: By definition and change of variables argument, we have

$$\begin{aligned} \| T_t f \|^p_{H^p(\mathbf{D})} &= \int_0^{2\pi} | (T_t f) e^{i\theta} |^p \, d\theta \\ &= \int_0^{2\pi} | f(\varphi_t(e^{i\theta})) | \varphi_t'(e^{i\theta}) i e^{i\theta} |^p \, d\theta. \end{aligned}$$

Let $\omega = \varphi_t(e^{i\theta})$, then $d\omega = \varphi_t'(e^{i\theta}) i e^{i\theta} d\theta$ and

$$\begin{aligned} \| T_t f \|^p_{H^p(\mathbf{D})} &= \int_0^{2\pi} | f(\omega) |^p \, d\omega \\ &= \| f \|^p_{H^p(\mathbf{D})} . \end{aligned}$$

This means that T_t is an isometry. Moreover, $T_t o T_s = T_{t+s}$ for all $s, t \in \mathbf{R}$ and $T_o = I$ where I is the identity operator. So $\{(T_t)\}_{t\in\mathbf{R}}$ is a group of isometries as desired.

To show that $\{(T_t)\}_{t\geq 0}$ is strongly continuous, it suffices to show that $\lim_{t\to 0} \| T_t f - f \|_p = 0$ for every $f \in H^p(\mathbf{D})$. Let $X(\mathbf{D})$ be the set containing all functions in $H^p(\mathbf{D})$ that are continuous on $\mathbf{D}$. Then $X(\mathbf{D})$ is dense in $H^p(\mathbf{D})$. Thus for $f \in H^p$ and arbitrary $\epsilon > 0$, there exists $g \in X(\mathbf{D})$ such that $\| f - g \|_p < \epsilon$, then

$$\| T_t f - f \|_p \leq \| T_t f - T_t g \|_p + \| T_t g - g \|_p + \| g - f \|_p$$

$$= 2 \| f - g \|_p + \| T_t g - g \|_p$$

Now for all $g \in X(\mathbf{D})$, $T_t g(z) \to g(z)$ for all $g \in \partial D$ and by isometry of (T_t), we have $\| T_t g \|_p \to \| g \|_p$ Fatous lemma then gives $\| T_t g - g \|_p \to 0$. Thus $\| T_t f - f \|_p \leq 2\epsilon$, and hence (T_t) is strongly continuous.

By definition, the infinitesimal generator H of T_t is given by

$$\begin{aligned} H(f) &= \lim_{t\to 0} \frac{T_t f - f}{t}, \ f \in D(H) \\ &= \lim_{n\to\infty} \frac{e^{-ct\gamma} f(e^{-ct}z) - f(z)}{t} \\ &= \frac{\partial}{\partial t}\left(e^{-ct\gamma} f(e^{-ct}z)\right)|_{t=0} \\ &= -c\gamma e^{-ct\gamma} f(e^{-ct}z) + e^{-ct\gamma} - ce^{-ct} z f'(e^{-ct}z) \\ &= -c\gamma f(z) - czf'(z), \end{aligned}$$

which implies that $D(H) \subseteq \{f \in H^p(\mathbf{D}) : zf'(z) \in H^p(\mathbf{D})\}$. Conversely, let $f \in H^p(D)$ such that $zf'(z) \in H^p(\mathbf{D})$. Then for $z \in \mathbf{D}$, we have

$$\begin{aligned} T_t f(z) - f(z) &= \int_0^t \frac{\partial}{\partial s}\left(e^{-cs\gamma} f(\varphi_s(z))\right) ds \\ &= \int_0^t \Big(-c\gamma e^{-ctH} f(\varphi_s(z)) - czef'(e^{-ct}z)\Big) \\ &= \int_0^t T_s(F) ds \end{aligned}$$

where $F(z) = -c\gamma f(z) - czf'(z)$. Thus $\lim_{t\to 0} \frac{T_t f - f}{t} = \lim_{t\to 0} \frac{1}{t} \int_o^t T_s(F) ds$ Now for $F \in H^p(\mathbf{D})$ the limit exists and equal to F. Thus $D(H) \supseteq \{f \in H^p(D) : zf'(z) \in H^p(D)\}$, as claimed.

To obtain the point spectrum of H, let λ be an eigenvalue and f be the corresponding eigenvector. Then the eigenvalue equation $Hf = \lambda f$ reduces to the differential equation

$$-cHf(z) - czf'(z) = \lambda f(z)$$

which is equivalent to

$$-czf'(z) = (\lambda + cH) f(z)$$

To solve the above ODE, we continue as follows;

$$\frac{f'(z)}{f(z)} = -\frac{1}{c}(\lambda + cH)\frac{1}{z}$$

$\Leftrightarrow$

$$\frac{df(z)}{f(z)} = -\frac{1}{c}(\lambda + cH)\frac{dz}{z}.$$

Therefore

$$\ln f(z) = -\frac{1}{c}(\lambda + cH)\ln z + C$$

and thus

$$f(z) = z^{-\frac{1}{c}(\lambda+cH)}$$

for $c \neq 0$. Since $z^{-\frac{1}{c}(\lambda+c\gamma)} \in H(\mathbf{D})$ if and only if $-\frac{1}{c}(\lambda + c\gamma) \in \mathbf{Z}_+$. That is $-(\gamma + \frac{\lambda}{c}) = n$, $n = 0, 1, 2, ...$ Hence $\sigma_p(H) = \{-c(n + \gamma) : n = 0, 1, 2, ...\}$

Clearly, if $Re(c) = 0$, then $c \in \rho(H)$ and therefore, the resolvent operator $(c - H)^{-1}$ reduces to

$$R(c, H)f(z) \quad = \quad \frac{1}{cz}\int_0^z f(\xi)d\xi.$$

As remarked by Cowen and Macluer [5], such resolvents are compact and therefore

$$\sigma(H) \quad = \quad \sigma_p(H)$$

Now by Theorem 5, $R(c, H)$ is decomposable and hence of scalar type

4 Conclusion

In this study we gave application of scalar type operators to Decomposibility. In particular, we showed that if H is of $(\alpha, \alpha + 1)$type $\mathbf{R}$ and that it generates a strongly continuous group on a Banach space, then its resolvent is Decomposable and therefore it is scalar type.

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Competing Interests

Author has declared that no competing interests exist.

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