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Action of $PGL(2, q)$ on the Cosets of the Centralizer of an Eliptic Element

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

Most researchers consider the action of projective general group on the cosets of its maximal subgroups leaving out non-maximal subgroups. In this paper, we consider the action of $PGL(2, q)$ centralizer of an elliptic element which is a non maximal subgroup C_{q+1}. In particular, we determine the subdegrees, rank and properties of the suborbital graphs of the action. We achieve this through the application of the action of a group by conjugation. We have proved that the rank is q and the subdegrees are $[1]^{[2]}$ and $[q + 1]^{[q-2]}$.

Keywords: Rank; subdegrees; centralizer; suborbital graphs.

1 Introduction

Let a group G act transitively on a set X. The orbits of the stabilizer G_α of a point $\alpha \in X$ are called *suborbits* of G on X. The number $R(G)$ of these suborbits is known as the *rank* of G on X and the suborbits length is known as the *subgegrees* of G on X. Rank and subdegrees are independent of the $\alpha \in X$ chosen. Any group G acts

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transitively on the set of right cosets of any of its subgroup. In this paper the set X is the set of the right cosets of $H = C_{q+1}$.

2 Preliminary Results

2.1 Background information on subgroup structure of $G = PGL(2, q)$ or $PSL(2, q)$

The general linear group, $GL(2, q)$ is a group of all 2×2 invertible matrices over a finite field, $GF(q)$, where $q = p^f$ for some prime p and $f \in \mathbb{Z}^+$. A subgroup, $SL(2, q)$ of $GL(2, q)$ consisting of all unimodular matrices is called a special linear group. The quotient groups of $GL(2, q)$ and $SL(2, q)$ by their centres are called the projective general linear group $PGL(2, q)$ and projective special linear group $PSL(2, q)$ respectively. The group $PGL(2, q)$ can be viewed as a group of linear fractional transformations of the form;

$$x \mapsto \frac{ax+b}{cx+d}, \tag{1}$$

where $a, b, c, d \in GF(q)$ and $ad - bc \neq 0$. (See [1] Sec 239)

The group $PSL(2, q)$ can be viewed as a group of linear fractional transformations of the form in Expression 1 with $ad - bc = 1$. (See [1]). The order of $PGL(2, q)$ is $q(q^2 - 1)$ and that of $PSL(2, q)$ is $\frac{q(q^2-1)}{k}$, where k is the gcd of 2 and $q - 1$. It follows that, if q is even, then $PGL(2, q)$ and $PSL(2, q)$ have the same order and since $PSL(2, q)$ $PGL(2, q)$, the two are isomorphic.

According to [1], G acts doubly transitively on the projective line $PG(1, q) = GF(q) \cup \{\infty\}$. Each non-identity element fixes either 0, 1 or 2 elements on $PG(1, q)$. Therefore the set of non-identity elements in G is partitioned into: τ_0, those elements that do not fix any element on $PG(1, q)$; τ_1, those elements that fix only one element on $PG(1, q)$ and τ_2, those elements that fix only 2 elements on $PG(1, q)$. The elements in the sets τ_0, τ_1 and τ_2 are known as *elliptic*, *parabolic* and *hyperbolic* elements of G respectively. (See [2] Chap 8)

Let $g \in \tau_2$ with $|g| \neq 2$. Then $C_G(g)$ is a cyclic subgroup C_{q-1} for $G = PGL(2, q)$ and $C_{\frac{q-1}{k}}$ for $G = PSL(2, q)$. These subgroups consist of identity and all hyperbolic elements of G fixing the same elements on $PG(1, q)$ as g. There are are $q(q + 1)$ such cyclic subgroups in G which only intersect pairwise at identity. It follows that τ_2 has $q(q + 1)(q - 2)$ and $q(q + 1)(\frac{q-1}{k} - 1)$ for $PGL(2, q)$ and $PSL(2, q)$ respectively. The normalizer of C_{q-1} in $PGL(2, q)$ is a dihedral subgroup $D_{2(q-1)}$ while that of $C_{\frac{q-1}{k}}$ in $PSL(2, q)$ is a dihedral subgroup $D_{2\frac{q-1}{k}}$. If $g \in \tau_2$ with $|g| = 2$, then $C_G(g)$ is a dihedral subgroup $D_{2(q-1)}$ and $D_{2\frac{q-1}{k}}$ for $PGL(2, q)$ and $PSL(2, q)$ respectively. (See [1] Sec 242)

If $g \in \tau_1$, then $C_G(g)$ is an Elementally Abelian group P_q of order q consisting of identity and parabolic elements fixing the same element on $PG(1, q)$ as g. $|PG(1, q)| = q + 1$ and therefore G has $q + 1$ such Elementally Abelian groups which only intersect pairwise at identity. Therefore $|\tau_1| = q^2 - 1$. It follows that all parabolic elements in $PGL(2, q)$ are conjugate in $PGL(2, q)$ but exists in two conjugacy classes in $PSL(2, q)$ of length $\frac{q^2-1}{2}$ each if q is odd. The normalizer of P_q in G is a subgroup of the form $P_q \ltimes C_{q-1}$ and $P_q \ltimes C_{\frac{q-1}{k}}$ for $PGL(2, q)$ and $PSL(2, q)$ respectively. The subgroup $N_G(P_q)$ consists P_q and hyperbolic elements in G whose one of the fixed point on $PG(1, q)$ is the same element fixed by P_q. (See [1] Sec 241)

Let $g \in \tau_0$ with $|g| \neq 2$. Then $C_G(g)$ is a cyclic subgroup C_{q+1} for $G = PGL(2, q)$ and $C_{\frac{q+1}{k}}$ for $G = PSL(2, q)$. These subgroups consist of identity and some elliptic elements of G. The normalizer of C_{q+1} in $PGL(2, q)$ is a dihedral subgroup $D_{2(q+1)}$ while that of $C_{\frac{q+1}{k}}$ in $PSL(2, q)$ is a dihedral subgroup $D_{2\frac{q+1}{k}}$. There are $q(q - 1)$ such cyclic groups in G which only intersect pairwise at identity. It follows that τ_2 has $q^2(q - 1)$ and $q(q - 1)(\frac{q+1}{k} - 1)$ for $PGL(2, q)$ and $PSL(2, q)$ respectively. If $g \in \tau_0$ with $|g| = 2$, then $C_G(g)$ is a dihedral subgroup $D_{2(q+1)}$ and $D_{2\frac{q+1}{k}}$ for $PGL(2, q)$ and $PSL(2, q)$ respectively. (See [1] Sec 243)

A summary of the subgroup structure G is also found in [3,4] and [5].

Theorem 1 *[6, p.9] Let G be a permutation group and $g, h \in G$. Suppose h has a cycle $(h_1\ h_2\ \cdots\ h_k)$, then $ghg^{-1} = (gh_1\ gh_2\ \cdots\ gh_k)$.*

Theorem 2 *[7] Let G be a group acting on set X. Then,*

$$|Orb_G(\alpha)| = \frac{|G|}{|G_\alpha|}. \tag{2}$$

Theorem 3 *[7] Let G act transitively on X. Then this action is equivalent to the action of G on the right cosets of G_α for $\alpha \in X$.*

Theorem 4 *A graph Γ is Eulerian if and only if each vertex has an even degree.*

Theorem 5 *[8, P. 44] Let G act transitively on set X. Then the number $R^*(G)$ of self-paired suborbits is given by,*

$$R^*(G) = \frac{1}{|G|}\sum_{g\in G} |fix(g^2)| \tag{3}$$

Theorem 6 *[9] Let G act on X and Γ be a suborbital graph corresponding to $\Delta(x)$. The following statements are equivalent.*

i. Γ is disconnected.
ii. The set $\mathcal{B}(x)$ of all the elements which are at a finite distance from x in Γ is a system of imprimitivity for Γ.
iii. There exists a system of imprimitivity $\daleth$ for Γ such that $x \in \daleth$ and $\daleth \cap \Delta(x) \neq \emptyset$.
iv. There exists a proper subgroup H of G containing G_x and $Orb_H(x) \cap \Delta(x) \neq \emptyset$.

Theorem 7 *[9] A group G acts primitively on X if and only if all the non-trivial suborbital graphs corresponding to the action are connected.*

Theorem 8 *[9] A group G act primitively X if and only if G_x for $x \in X$ is a maximal subgroup of G.*

Theorem 9 *[9] If Γ is a disconnected suborbital graph, then the set of vertices in the component containing x is $Orb_H(x)$ where H is a subgroup of G containing G_x and $Orb_H(x) \cap \Delta(x) \neq \emptyset$.*

Theorem 10 *[10] Let Γ be a suborbital graph of a transitive action. Then all disconnected components of Γ are isomorphic.*

Proposition 2.1 *Let G act transitively on X and let Γ_i be any undirected suborbital graph corresponding to $\Delta_i(x)$, where $x \in X$. Then the number $\lambda(\Gamma_i)$ of triangles in Γ_i is given by,*

$$\lambda(\Gamma_i) = \frac{|X||\Delta_i(x)\cap\Delta_i(y)||\Delta_i(x)|}{6} \quad \text{where } y \in \Delta_i(x). \tag{4}$$

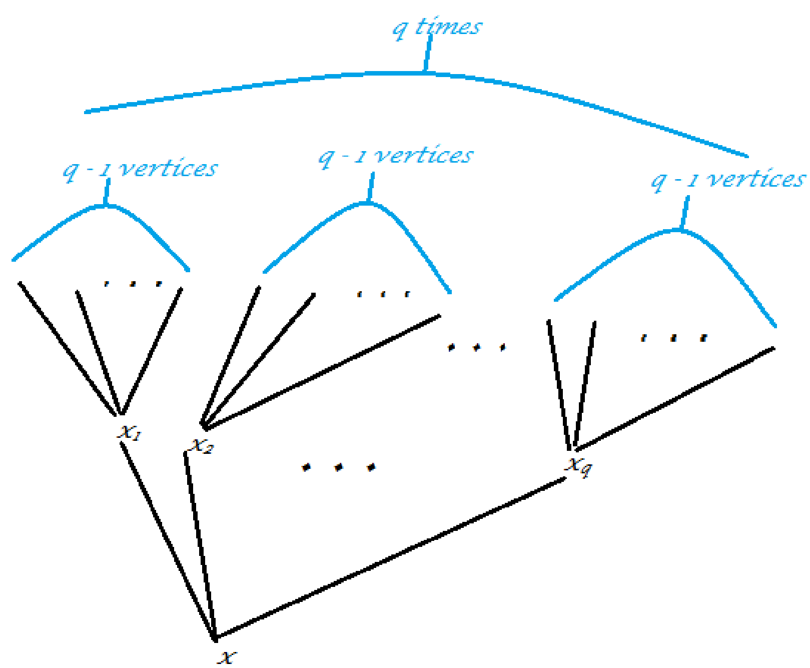
Proof. All the edges of Γ_i originating from vertex $x \in X$ are of the form (x, y) where $y \in \Delta_i(x)$. A triangle in Γ_i with (x, y) as an edge is formed by vertices x, y and z such that $z \in \Delta_i(x) \cap \Delta_i(y)$. The number of such z and y is $|\Delta_i(x) \cap \Delta_i(y)|$ and $|\Delta_i(x)|$ respectively. The number $|\Delta_i(x) \cap \Delta_i(y)|$ is independent of x and y chosen.[See [9]] Therefore $|\Delta_i(x) \cap \Delta_i(y)||\Delta_i(x)|$ is twice the number of triangles with x as a vertex. Since Γ_i is regular, this number is same for any $x \in X$. A triangle has 3 vertices and therefore $|X||\Delta_i(x) \cap \Delta_i(y)||\Delta_i(x)|$ is 6 times the number of triangles in Γ_i. Hence we obtain Equation 4.

Proposition 2.2 *Let G act transitively on X. Then any suborbital graph corresponding to a non-trivial self-paired suborbit of G of length one is a forest and has $\frac{|X|}{2}$ components.*

Proof. The suborbital graphs of a transitive action are regular and have valency equal to the length of the corresponding suborbit. In this case the length is one and therefore each vertex is adjacent to only one vertex. Therefore the graph is disconnected and each component has two vertices. The graph is a forest since there are no non-trivial circuits. It follows that the graph has $\frac{|X|}{2}$ components.

Proposition 2.3 *Let Γ_i be a suborbital graph of G acting transitively on X corresponding to a self-paired suborbit $\Delta_i(x)$ of length r. If $r^2 + 1 > |X|$, then the girth of Γ_i is less than or equal to 4.*

Proof. Let Γ_i be the suborbital graph corresponding to $\Delta_i(x)$ with $|\Delta_i(x)| = q$. Let $\{x_1, x_2, \dots, x_q\} = \Delta_i(x)$. Γ_i has undirected edges (x, x_j) with $j = 1,2, \dots, q$. If $\Delta_i(x) \cap \Delta_i(x_j) \neq \emptyset$ then Γ_i has a triangle formed by $x \to x_j \to v \to x$ where $v \in \Delta_i(x) \cap \Delta_i(x_j)$. Suppose $\Delta_i(x) \cap \Delta_i(x_j) = \emptyset \ \forall j = 1,2, \dots, q$. Then Γ_i has no triangles. We need to show that $\left(\Delta_i(x_j) \cap \Delta_i(x_k)\right) - \{x\} \neq \emptyset$ for some $j, k = 1,2, \dots, q$ with $j \neq k$. By way of contradiction suppose $\left(\Delta_i(x_j) \cap \Delta_i(x_k)\right) - \{x\} = \emptyset$ for some $j, k = 1,2, \dots, q$ with $j \neq k$. Then each of $x_j \in \Delta_i(x)$ with $j = 1,2, \dots, q$ is adjacent to other $q - 1$ vertices different from vertices in $\{x\} \cup \Delta_i(x)$. These vertices are $q(q-1)$. Adding the vertex x and q vertices in $\Delta_i(x)$ we get $q^2 + 1$ vertices which is greater than the vertices in Γ_i. This leads to a contradiction and therefore $\left(\Delta_i(x_j) \cap \Delta_i(x_k)\right) - \{x\} \neq \emptyset$. Hence there exists a cycle of length 4. This cycle is formed by vetices $x \to x_j \to v \to x_k \to x$,where $v \in \left(\Delta_i(x_j) \cap \Delta_i(x_k)\right) - \{x\}$.



3 Main Results

3.1 Subdegrees of $G = PGL(2, q)$ acting on the cosets of $H = C_{q+1}$

Lemma 11 *The action of G on the cosets of H is equivalent to the action of G on the set C^x where $x \in \tau_0$ and $|x| > 2$.*

Proof. The subgroup $Stab_G$(x) is H. By Theorem 3, the two actions are equivalent.

Theorem 12 *Let H act on cosets of a subgroup C_{q+1}. Then the rank is q and the subdegrees are* $[1]^{[2]}$ *and* $[q+1]^{[q-2]}$.

Proof. By Lemma 11, the action is equivalent to the action of G on C^x where $x \in \tau_0$ with $|x| > 2$. In $H = C_{q+1}$ there are only two elements of C^x. The centralizer of each of these elements is H. Therefore by Theorem 2, each of the two elements is self-conjugate in G and is contained in a suborbit of length 1. Thus there are two suborbits of length 1. If $y \in (C^x - H)$, then its centralizer in H is $\{I\}$. Therefore, by Theorem 2, $|Orb_H(\mathrm{x})| = q+1$. Since $|C^x - H| = q^2 - q - 2$, there are $q-2$ suborbits of length $q+1$. Summing these suborbits, we obtain the rank is q.

4 Suborbital Graphs of $G = PGL(2,q)$ Acting on the Cosets of $H = C_{q+1}$

Theorem 13 *All suborbits of $PGL(2,q)$ acting on the cosets of C_{q+1} are self-paired.*

Proof. By Lemma 11 this action is equivalent to the action of G on C^x, where $x \in \tau_0$ and $|x| \neq 2$ by conjugation. The square of each of q^2 involutions in G and I^2 fixes $q(\mathrm{q}-1)$ elements in C^x. If g is an elliptic element with $|g| > 2$, then g^2 fixes only h and h^{-1} in C^x such that $h \in C_{q+1}$ containing g. If g is either a hyperbolic or a parabolic element with $|g| > 2$, then g^2 does not fix any element in C^x. Therefore, if q is odd, the number of self-paired suborbits $R^*(G)$ is given by,

$$R^*(G) = \frac{1}{q(q^2-1)}[q(q-1) + q(q-1)q^2 + (q-1)\frac{q(q-1)}{2}] = q. \quad (5)$$

If q is even,

$$R^*(G) = \frac{1}{q(q^2-1)}[q(q-1) + q(q-1)(q^2-1) + q(\frac{q(q-1)}{2}] = q. \quad (6)$$

From Theorem 12, the rank is q. By Equations 5 and 6, it follows that all suborbits are self-paired.

Theorem 14 *Let Γ be a suborbital graph of G acting on the cosets of H corresponding to a suborbit $\Delta(x)$ of length* $q+1$. *Then Γ is connected.*

Proof. There is no proper subgroup F of G properly G_x where x is an elliptic element of order $q+1$ such that $Orb_F(x) \cap \Delta(x) \neq \emptyset$. Therefore, by Theorem 6, Γ is connected.

Theorem 15 *Let Γ be a suborbital graph corresponding to a suborbit of length* $q+1$. *Then the girth of Γ is either 3 or 4.*

Proof. There are $q^2 - q$ cosets. Also $(q+1)^2 + 1 = q^2 + 2q + 2 > q^2 - q$. Therefore by Proposition 2.3, Γ has girth 3 or 4.

Theorem 16 *The suborbital graph Γ corresponding to the non-trivial suborbit of length 1 is a forest with* $\frac{q^2-q}{2}$ *components.*

Proof. The result follows from Proposition 2.2 and Theorem 13.

Theorem 17 *Let G act on the cosets of H, where q is odd. Then all suborbital graphs corresponding to suborbits of length* $q+1$ *are Eulerian.*

Proof. The vertex degree of these graphs is $q+1$ and it is even. Therefore the graphs are Eulerian.
We now give an algorithm to construct the suborbital graphs of G acting on the cosets of H when q 3. In the algorithm, we first express G as a permutation group on $PG(1,q)$ and then compute the permutation group of G acting on C^g, where $|g| = q+1$.

Algorithm 4.1 *The following steps are used to construct the suborbital graphs corresponding to this action when q 3.*

1. Find the elements of of $PG(1,q)$ and index them as follows where α is the generator of $GF(q)^*$: $1:=\alpha$, $2:=\alpha^2$,..., $q-1:=\alpha^{q-1}$, $q:=0$, $q+1:=\infty$.
2. Find the generators of G using $g_1=\alpha w$, $g_2=\frac{1}{w}$ and $g_3=w+1$ as permutations on $PG(1,q)$. g_1 takes the form $(1\ 2\ \dots\ q-1)$, g_2 takes the form $(1\ q-2)(2\ q-3)\dots(\frac{q-3}{2}\ \frac{q+1}{2})(q\quad q+1)$ if q is odd and $(1\ q-2)(2\ q-3)\dots(\frac{q-2}{2}\ \frac{q}{2})(q\ q+1)$ if q is even. g_3 depends on the generator of $GF(q)^*$ chosen.
3. Find an element of order $q+1$ by multiplying permutations above and call it g_4. Now $G=<g_1,g_2,g_3>$ $=<g_1,g_4>$.
4. Find the conjugates of g_4 in G using g_1, g_2, g_3 and g_4 using Theorem 1. Index them according to the way g_4 permutes them. As a result g_1', g_2', g_3' and g_4' will be obtained.
5. Obtain the suborbits from g'_4. They correspond to the disjoint cycles of g'_4.
6. Let $\Delta_i(x)$ be the i^{th} G_x −orbit. Then set $\Delta_i(x)$ contains the vertices adjacent to x. The vertices adjacent to gx are the set $\Delta_i(gx)=g\Delta_i(x)$. By applying g'_1 and g'_4 to $\Delta_i(x)$ and x, all edges of Γ_i can be obtained as $(\eta,f(\eta))$, where
$$f(\eta)=\begin{cases} y\in g'_4\Delta_i(\eta-1), & \text{if}\eta-1\text{isnotthelastelementinacycleofg}'_4 \\ y\in\Delta_i(\xi):(g'_1)^m\xi=\eta, & \text{if}\eta-1\text{isthelastelementinacycleofg}'_4 \end{cases}, \quad (7)$$
where $m\in\mathbb{N}$. Since the all suborbits are self-paired, $(f(\eta),\eta)$ is also an edge of Γ_i.
7. Plot $\frac{q(q-1)}{2}$ vertices and edges obtained as in 4.1.

Example: Construction suborbital graphs of $G=PGL(2,5)$ acting on the cosets of C_6

A generator of $\mathbb{Z}_5^*$ is 2. We now index the elements of $PG(1,5)$.

$1:=2$, $2:=2^2=4$, $3:=2^2=3$, $4:=2^4=1$, $5:=0$, $6:=\infty$.

In this case $g_1=(1\ 2\ 3\ 4)$, $g_2=(1\ 3)(5\ 6)$ and $g_3=(1\ 3\ 2\ 5\ 4)$. By trial and error method we find g_4 through multiplications of g_1 g_2 and g_3. We get $g_4=g_3g_2g_1=(1\ 5\ 6\ 4\ 2\ 3)$ and $g_4^{-1}=(1\ 3\ 2\ 4\ 6\ 5)$. Now $G=\langle g_1,g_4\rangle$.

Next we find g'_4 by applying g_4 on $(1\ 5\ 6\ 4\ 2\ 3)$.

$(1:=(1\ 5\ 6\ 4\ 2\ 3))(2:=(1\ 3\ 4\ 2\ 5\ 6)\ 3:=(1\ 2\ 3\ 6\ 4\ 5)\ 4:=(1\ 4\ 2\ 6\ 5\ 3)\ 5:=(1\ 5\ 2\ 3\ 4\ 6)\ 6:=(1\ 2\ 4\ 5\ 6\ 3)\ 7:=(1\ 5\ 3\ 2\ 6\ 4))\ (8:=(1\ 6\ 2\ 5\ 3\ 4)\ 9:=(1\ 2\ 5\ 4\ 3\ 6)\ 10:=(1\ 4\ 5\ 3\ 6\ 2)\ 11:=(1\ 4\ 3\ 5\ 2\ 6)\ 12:=(1\ 6\ 3\ 4\ 5\ 2)\ 13:=(1\ 2\ 6\ 3\ 5\ 4))(14:=(1\ 3\ 2\ 4\ 6\ 5))(15:=(1\ 6\ 5\ 3\ 4\ 3)\ 16:=(1\ 5\ 4\ 6\ 3\ 2)\ 17:=(1\ 3\ 5\ 6\ 2\ 4)\ 18:=(1\ 6\ 4\ 3\ 2\ 5)\ 19:=(1\ 3\ 6\ 5\ 4\ 2)\ 20:=(1\ 4\ 6\ 2\ 3\ 5))$

Therefore $g'_4=(1)(2\ 3\ 4\ 5\ 6\ 7)(8\ 9\ 10\ 11\ 12\ 13)(14)(15\ 16\ 17\ 18\ 19\ 20)$.

Applying g_1

$(1:=(1\ 5\ 6\ 2\ 4\ 3)\ 2:=(1\ 3\ 4\ 2\ 5\ 6)\ 17:=(1\ 3\ 5\ 6\ 2\ 4)\ 6:=(1\ 2\ 4\ 5\ 6\ 3))\ (5:=(1\ 5\ 2\ 3\ 4\ 6)\ (8:=(1\ 6\ 2\ 5\ 3\ 4)\ 13:=(1\ 2\ 6\ 3\ 5\ 4)\ 3:=(1\ 2\ 3\ 6\ 4\ 5))(14:=(1\ 3\ 2\ 4\ 6\ 5)\ 15:=(1\ 6\ 5\ 3\ 4\ 3)\ 4:=(1\ 4\ 2\ 6\ 5\ 3)\ 19:=(1\ 3\ 6\ 5\ 4\ 2))(18:=(1\ 6\ 4\ 3\ 2\ 5)\ 11:=(1\ 4\ 3\ 5\ 2\ 6)\ 10:=(1\ 4\ 5\ 3\ 6\ 2)\ 16:=(1\ 5\ 4\ 6\ 3\ 2))\ (7:=(1\ 5\ 3\ 2\ 6\ 4)\ 9:=(1\ 2\ 5\ 4\ 3\ 6)\ 20:=(1\ 4\ 6\ 2\ 3\ 5)\ 12:=(1\ 6\ 3\ 4\ 5\ 2))$

Therefore $g'_1=(1\ 2\ 17\ 6)(5\ 8\ 13\ 3)(14\ 15\ 4\ 19)(18\ 11\ 10\ 16)(7\ 9\ 20\ 12)$.
The suborbits are can be extracted from the disjoint cycles of g'_4 and they are $\Delta_0(1)=\{1\}$, $\Delta_1(1)=\{2,3,4,5,6,7\}$, $\Delta_2(1)=\{8,9,10,11,12,13\}$, $\Delta_3(1)=\{14\}$ and $\Delta_5(1)=\{15,16,17,18,19,20\}$.

Using these suborbits and Equation 7, we obtain Array 8.

$\Delta_1(1) = \{2,3,4,5,6,7\}$	$\Delta_2(1) = \{8,9,10,11,12,13\}$	$\Delta_3(1) = \{14\}$	$\Delta_4(1) = \{15,16,17,18,19,20\}$
$\Delta_1(2) = \{17,5,19,8,1,9\}$	$\Delta_2(2) = \{13,20,16,10,7,3\}$	$\Delta_3(2) = \{15\}$	$\Delta_4(2) = \{4,18,6,11,14,12\}$
$\Delta_1(3) = \{18,6,20,9,1,10\}$	$\Delta_2(3) = \{8,15,17,11,2,4\}$	$\Delta_3(3) = \{16\}$	$\Delta_4(3) = \{5,19,7,12,14,13\}$
$\Delta_1(4) = \{19,7,15,10,1,11\}$	$\Delta_2(4) = \{9,16,18,12,3,5\}$	$\Delta_3(4) = \{17\}$	$\Delta_4(4) = \{6,20,2,13,14,8\}$
$\Delta_1(5) = \{20,2,16,11,1,12\}$	$\Delta_2(5) = \{10,17,19,13,4,6\}$	$\Delta_3(5) = \{18\}$	$\Delta_4(5) = \{7,15,3,18,14,10\}$
$\Delta_1(6) = \{15,3,17,12,1,13\}$	$\Delta_2(6) = \{11,18,20,8,5,7\}$	$\Delta_3(6) = \{19\}$	$\Delta_4(6) = \{2,16,4,9,14,10\}$
$\Delta_1(7) = \{16,4,18,13,1,14\}$	$\Delta_2(7) = \{12,19,15,9,6,2\}$	$\Delta_3(7) = \{20\}$	$\Delta_4(7) = \{3,17,5,10,14,11\}$
$\Delta_1(8) = \{12,17,18,10,2,7\}$	$\Delta_2(8) = \{16,6,14,3,19,1\}$	$\Delta_3(8) = \{11\}$	$\Delta_4(8) = \{9,4,5,13,15,20\}$
$\Delta_1(9) = \{13,18,19,11,3,2\}$	$\Delta_2(9) = \{17,7,14,4,20,1\}$	$\Delta_3(9) = \{12\}$	$\Delta_4(9) = \{10,5,6,8,16,15\}$
$\Delta_1(10) = \{8,19,20,12,4,3\}$	$\Delta_2(10) = \{18,2,14,5,15,1\}$	$\Delta_3(10) = \{13\}$	$\Delta_4(10) = \{11,6,7,9,17,16\}$
$\Delta_1(11) = \{9,20,15,13,5,4\}$	$\Delta_2(11) = \{19,3,14,6,16,1\}$	$\Delta_3(11) = \{8\}$	$\Delta_4(11) = \{12,7,2,10,18,17\}$
$\Delta_1(12) = \{10,15,16,8,6,5\}$	$\Delta_2(12) = \{20,4,14,7,17,1\}$	$\Delta_3(12) = \{9\}$	$\Delta_4(12) = \{11,16,17,9,7,6\}$
$\Delta_1(13) = \{11,16,17,9,7,6\}$	$\Delta_2(13) = \{15,5,14,8,18,1\}$	$\Delta_3(13) = \{10\}$	$\Delta_4(13) = \{12,17,10,8,7\}$
$\Delta_1(14) = \{15,20,19,18,17\}$	$\Delta_2(14) = \{12,11,10,9,8,13\}$	$\Delta_3(14) = \{1\}$	$\Delta_4(14) = \{2,7,6,5,4,3\}$
$\Delta_1(15) = \{4,12,14,11,6,18\}$	$\Delta_2(15) = \{7,10,16,20,13,3\}$	$\Delta_3(15) = \{2\}$	$\Delta_4(15) = \{17,9,1,8,19,5\}$
$\Delta_1(16) = \{5,13,14,12,7,19\}$	$\Delta_2(16) = \{2,11,17,15,8,4\}$	$\Delta_3(16) = \{3\}$	$\Delta_4(16) = \{18,10,1,9,20,6\}$
$\Delta_1(17) = \{6,8,14,13,2\}$	$\Delta_2(17) = \{3,12,18,16,9,5\}$	$\Delta_3(17) = \{4\}$	$\Delta_4(17) = \{19,11,1,10,15,7\}$
$\Delta_1(18) = \{7,9,14,8,3,15\}$	$\Delta_2(18) = \{4,13,19,17,10,6\}$	$\Delta_3(18) = \{5\}$	$\Delta_4(18) = \{20,12,1,11,16,2\}$
$\Delta_1(19) = \{2,10,14,9,4,16\}$	$\Delta_2(19) = \{5,8,20,18,11,7\}$	$\Delta_3(19) = \{6\}$	$\Delta_4(19) = \{15,13,1,12,17,3\}$
$\Delta_1(20) = \{3,11,14,10,5,17\}$	$\Delta_2(20) = \{6,9,15,19,12,2\}$	$\Delta_3(20) = \{7\}$	$\Delta_4(20) = \{16,18,1,13,18,4\}$

(8)Using Array 8, we obtain Figures 1, 2, 3 and 4 representing suborbital graphs Γ_1, Γ_2, Γ_3 and Γ_4 corresponding to $\Delta_1(1)$, $\Delta_2(1)$, $\Delta_3(1)$ and $\Delta_4(1)$ respectively.

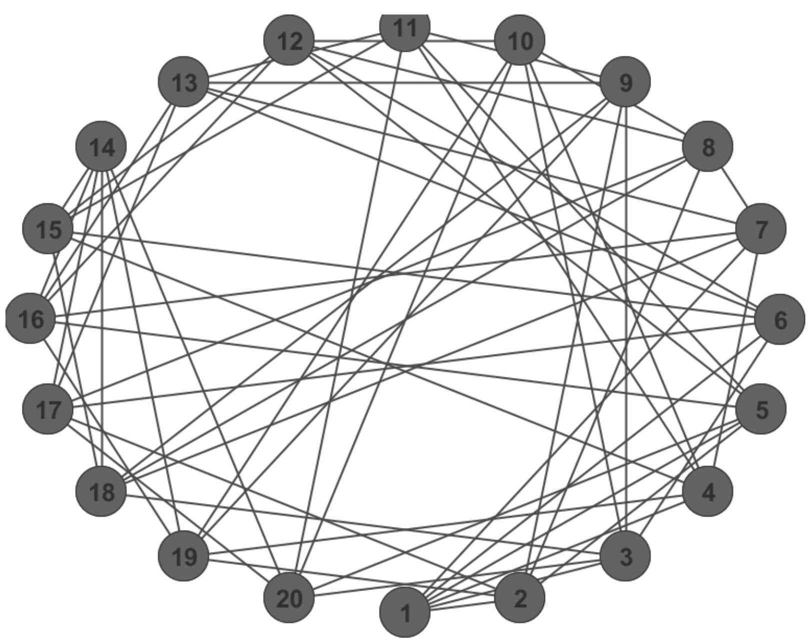


Fig. 1. Suborbital graph Γ_1

$|\Delta_1(1) \cap \Delta_1(2)| = 1$ and therefore, by Proposition 2.1, Γ_1 has 20 triangles. It follows that the girth is 3. The graph is also connected and has diameter 3.

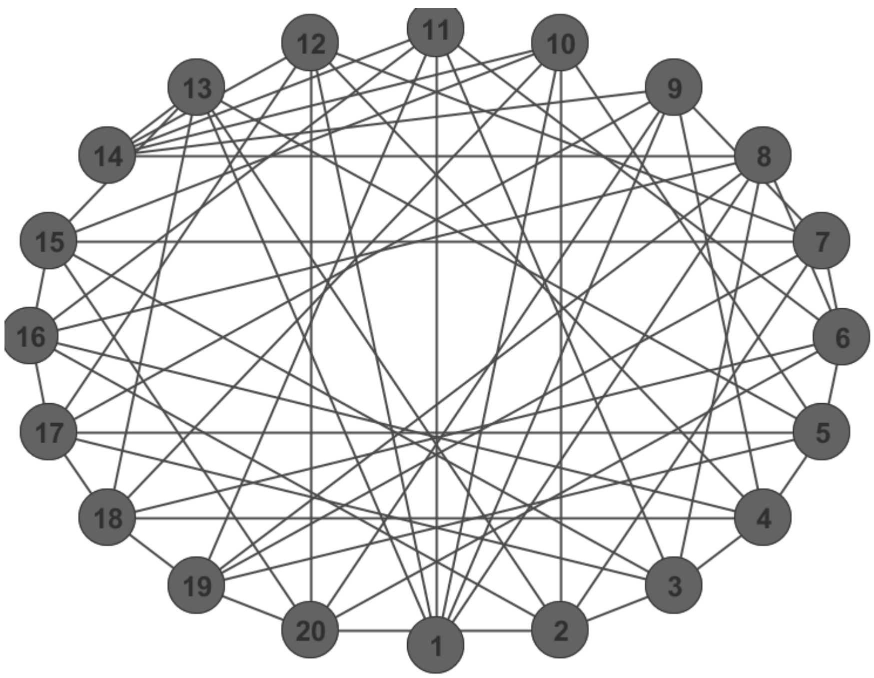


Fig. 2. Suborbital graph Γ_2

$|\Delta_2(1) \cap \Delta_2(2)| = 0$ and therefore, by Proposition 2.1, Γ_2 has no triangles. It follows that the girth is 4 by Theorem 15. The graph is also connected and has diameter 3.

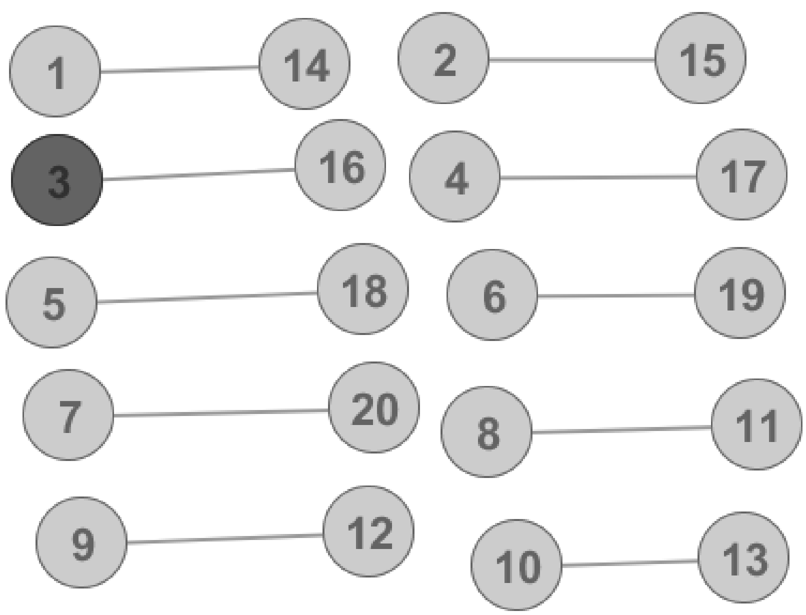


Fig. 3. Suborbital graph Γ_3

Γ_3 is disconnected with 10 components. The graph is a forest and therefore, the girth is 0. Note that each component of Γ_3 are of size 2 consisting of x and x^{-1} where $x \in C^{g_4}$.

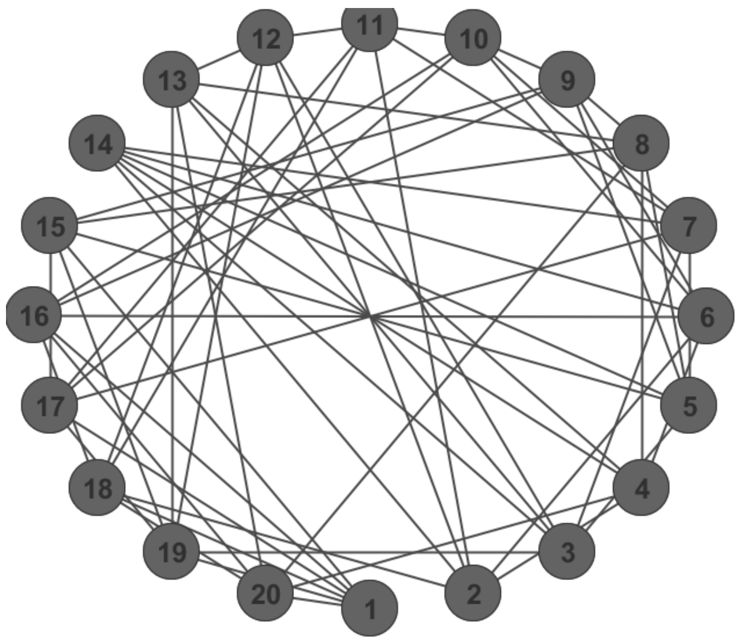


Fig. 4. Suborbital graph Γ_4

$|\Delta_4(1) \cap \Delta_4(2)| = 2$ and therefore, by Proposition 2.1, Γ_4 has 40 triangles. It follows that the girth is 3. The graph is also connected and has diameter 3.

Competing Interests

Authors have declared that no competing interests exist.

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