Nonlinear Schrodinger Equations with Variable Coefficients: Numerical Integration

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

The nonlinear Schrödinger equation in one-dimensional coordinate is investigated numerically by employing explicit and implicit finite difference methods. It is shown that the explicit method is conditionally stable, and the condition for stability, which is a function of the time step and spatial step size, or several grid points are obtained. It is also shown that the implicit scheme is unconditionally stable and results in a tridiagonal matrix at each time level as a function of the nonlinear term. The effects of dispersion and nonlinearity concerning the time and space steps are also investigated. The validity of our scheme is established by reproducing some existing results on the constant-coefficient nonlinear Schrödinger equation. The schemes are then extended to study the variable coefficient equation, which has a growing interest and applications in many areas of nonlinear science.

1 Introduction

Several authors have been greatly interested in the nonlinear Schrödinger (NLS) equation with constant coefficients and its variants in one-dimensional space. Many numerical methods such as finite difference, finite

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elements, and time-splitting spectral methods have been used to analyse the NLS equation in a finite domain subject to different boundary conditions. NLS has number of applications [1-7]]. Ivanauskas and Radziunas [8] investigated the NLS equation utilising finite difference methods with some grid analogues of the conservation law. They used a Dirichlet type of boundary condition on both grid boundaries. They proved that the difference scheme is stable and converges in the L^2-norm and established a grid conservation law for the difference scheme of DuFort-Frankel type. Ramos [9] employed three linearly implicit finite difference methods to study the NLS equation in the presence of damping and pumping. They analysed soliton propagation through a medium whose refractive index undergoes a smooth, finite change over a distance as a function of the jump in the refractive index and the distance over which the jump occurs. Fei, Pérez-García [10] constructed a linearly implicit Crank-Nicholson scheme for the NLS type equations and showed that their scheme conserves energy and charge of the system. They validated these through numerical experiments that confirm their scheme's stability, accuracy, and efficiency.

The variable coefficients evolution type equations have great applications due to many physical phenomena that capture this behaviour. But due to the complications involved in deriving the conservation for variable coefficients, previous authors have focused on the case where dispersion and nonlinearity coefficients are treated as constants. In this study, we explore numerical integration by extending these coefficients so that they depend on space and time. The NLS equation with cubic nonlinearity, whose coefficients depend on the evolution variable, was first predicted to exist in optical applications. It was later discovered that the same equations apply in the model of Bose-Einstein condensate [11]. The case where the coefficients are functions of the spatial coordinate has also found applications. The nonlinear Schrödinger (NLS) equation describes a broad class of physical phenomena; modulational instability of water waves, propagation of heat pulses in anharmonic crystals, the helical motion of a very thin vortex filament, nonlinear modulation of collision-less plasma waves, self-trapping of a light beam in a dispersive colour system [12]. In optics, the NLS equation occurs in the Manakov system, a wave propagation model in fibre optics. Indeed, the nonlinear Schrödinger equation is one of the most fundamental nonlinear lattice dynamical models. It has a wide range of applications; it is the relevant dispersive envelope wave model for describing the electric field in optical fibres, for the self-focusing and collapse of Langmuir waves in plasma physics, or for the description of freak waves (the so-called rogue waves) in the ocean [13]. The growing interest in wave propagation in nonlinear photonic band-gap materials and periodic nonlinear dielectric superlattices has also contributed a great deal to understanding the NSL equations [13].

The well-known solutions of the NLS equation are those for solitary waves or solitons. The theory of the NLS equation was first developed in 1971 [14]. They first applied the inverse scattering transform method to this equation and derived a more general form for bright and dark solitons. These efforts have gained a lot of popularity over the years. NLS equation solitons have been verified experimentally, and various intriguing properties of solitons derivable from the result of inverse scattering transform theory were identified. It was theoretically shown that an optical pulse in a dielectric fibre forms a solitary wave because the wave envelope satisfies the NLS equation [15]. Optical solitons were experimentally observed in 1980 [16].

Many other evolution equations such as time-fractional Benjamin-Bona Mahony equation (TFBBM) [17], (3 + 1)-dimensional extended date–Jimbo–Kashiwara–Miwa equation [18], Bruta-Gelfand equation [19], and similar boundary value problems arising from an adiabatic tubular chemical reactor theory [20], nuclear physics [21], and magneto-hydrodynamics incompressible nanofluid flow past over an infinite rotating disk [22] have been solved using different analytical and numerical approaches [17] applied the integrating factor property to obtain analytical solution of TFBBM equation after reducing the equation to nonlinear fractional ordinary differential equation using its Lie symmetry. In [18], a new exact Lump-soliton solution that localized in all spatio-temporal directions was derived using Hirota method. Other methods of solution include operational matrix differentiation with Newton Raphson technique [19], Hermite wavelets technique via Newton Raphson method [20], Taylor wavelets via Newton iterative technique, and method of lines via Runge-Kutta technique [22].

There have been various investigations on the numerical solutions of variants of the nonlinear Schrödinger equation based on either the finite difference [10,9,23,24]] and the finite element, the spectral methods. Most of these studies have been devoted to the NLS equation solutions that have a special solution with the form of a pulse, that is, solitons, keeping their shapes and velocities after an interaction. Such solutions are further investigated here. Moreover, in most applications of the NLS equation that model various physical phenomena, constant-coefficient type equations have been used. These involve some assumptions and approximations of the

natural system. A more realistic model could be achieved by considering variable coefficients type evolution equations. The introduction of variable coefficients makes the equation analytical intractable hence numerical approach has to be considered. What is of great interest to be simulated is how the variations of dispersion coefficient and nonlinear term affect the solitary wave solutions and how to maintain a balance of these variables.

In this study, explicit and implicit finite difference methods are investigated for the study of focusing, nonlinear Schrödinger equation with extension to variable coefficients in one-dimensional coordinate and finite domains subject to zero boundary conditions on both boundaries. Many authors have devoted efforts to applying these methods with tremendous success, but only considering the special case, where the coefficients are treated as constants. For instance, Leble and Reichel [23] have successfully used similar numerical schemes to investigate coupled nonlinear Schrödinger equations. In this work, the convergence and stability of these numerical methods are assessed as a function of the time step and spatial step size. The numerical calculations helped us verify the clear distinction between the two schemes. T

Furthermore, the effects of varying dispersion coefficients and nonlinear terms as the pulse propagates through a medium will be significant in studying nonlinear waves. This notion has a growing interest and numerous applications in many areas of nonlinear science. For instance, in a fibre-optic communication system [25,26]], information is transmitted over fibre using a coded sequence of optical pulses whose width is determined by the bit rate of the system. When the fibre introduces dispersion, it interferes with the transmission process by broadening the pulse, leading to error if it spreads beyond its bit rate. A technique used to cope with the dispersion-induced is called dispersion management. It is implemented by either changing the dispersion characteristics of the fibre or by introducing dispersion compensating fibre to cancel the dispersion built up as signal travel through the network.

2 The Governing Model

The NSL equation is given by

$$i\frac{\partial U}{\partial t} + \frac{k}{2}\frac{\partial^2 U^2}{\partial x^2} U + q|U|^2 U = 0, \tag{1}$$

where $U(x, t)$ is the slowly varying dispersive wave envelope propagating in a nonlinear medium, k is the dispersion constant, q is nonlinearity strength, the variable t is time, and x is the spatial coordinate [26]. The term $q|U|^2 U$ can be extended to the general form $q|U|^2 = f(|U|^2)$.

The function $f(|U|^{\{2\}})$ characterises the nonlinear medium, for example, the nonlinear correction of the photonic band-gap materials' refractive index or the quasi-particles self-interaction in the superlattices [27]. Equation (1) can be solved exactly by using the inverse scattering transform reference [28]. However, the resulting solutions are not very explicit except for the exceptional cases of soliton solutions. Furthermore, in real applications, the presence of forces or dissipations may introduce some perturbation to the actual model (1) and thereby make it analytically unsolvable. A good way of exploring the complicated perturbed soliton model is to employ numerical simulations. However, not all schemes that can give reliable numerical results must be taken in selecting numerical techniques. An inappropriate discretisation of the system may result in a ``blow-up'' [10]. We investigate numerical methods for solving the NSL equations based on the finite difference schemes. Such methods have been applied to model vector spatial solitons behaviour in nonlinear waveguide arrays. To start with, let us derive the conservation law for equation (1).

3 Conservation law

3.1 Constant coefficients: k, q are constants

There are an infinite number of conserved quantities for equation (1). We establish the conservation of energy (E), also known as the L^2-norm. The momentum (M) and the Hamiltonian (H) among other quantities are conserved.

We write the conjugate of equation (1) as follows:

$$-i\frac{\partial \bar{U}}{\partial t} + \frac{k}{2}\frac{\partial^2 \bar{U}}{\partial x^2} + q|U|^2\bar{U} = 0. \tag{2}$$

Multiplying equation (1) by $\bar{U}$ and equation (2) by $-U$, we obtain

$$i\bar{U}\frac{\partial U}{\partial t} + \frac{k\bar{U}}{2}\frac{\partial^2 U}{\partial x^2} + q|U|^2\bar{U}U = 0,$$
$$iU\frac{\partial \bar{U}}{\partial t} - \frac{kU}{2}\frac{\partial^2 \bar{U}}{\partial x^2} - q|U|^2 U\bar{U} = 0. \tag{3}$$

Adding the equations in (3) and noting that

$$U\bar{U} = \bar{U}U = |U|^2,$$

we obtain the following

$$i\frac{\partial(U\bar{U})}{\partial t} + \frac{k\bar{U}}{2}\frac{\partial^2 U}{\partial x^2} - \frac{kU}{2}\frac{\partial^2 \bar{U}}{\partial x^2} = 0, \tag{4}$$

and integrating the result, we arrive at

$$i\frac{\partial}{\partial t}\int_{-\infty}^{\infty}|U|^2\,dx + \int_{-\infty}^{\infty}\frac{k\bar{U}}{2}\frac{\partial^2 U}{\partial x^2}\,dx - \int_{-\infty}^{\infty}\frac{kU}{2}\frac{\partial^2 \bar{U}}{\partial x^2}\,dx = 0,$$
$$= i\frac{\partial}{\partial t}\int_{-\infty}^{\infty}|U|^2\,dx + \frac{k}{2}[\bar{U}\partial_x U - U\partial_x\bar{U}]_{-\infty}^{\infty}. \tag{5}$$

Suppose the function U vanishes on the boundary. We obtain

$$\lim_{x\to\pm\infty} U = 0.$$

Then

$$i\frac{\partial}{\partial t}\int_{-\infty}^{\infty}|U|^2\,dx = 0. \tag{6}$$

Therefore, we obtain the conservation law in the form

$$\int_{-\infty}^{\infty}|U|^2\,dx = const. \tag{7}$$

3.2 Variable coefficients: k, q are functions of x, t

These conservation laws hold for any function $q = q(x,t)$ and $k = k(t)$. But for k, we have some restrictions on the conservation laws to hold. Once k depends on space, we have difficulty integrating the last term of equation (5). Thus, the NLS equation with variable coefficients is given by

$$i\frac{\partial U}{\partial t} + \frac{k(t)}{2}\frac{\partial^2 U}{\partial x^2}U + q(x,t)|U|^2 U = 0, \tag{8}$$

where $k(t)$ represents the dispersion coefficient, which in communication is significant (and is used to model strong dispersion management) [29], and $q(x)$ is the loss-gain coefficient. The NLS equation with variable coefficients is used in the optical fibre to study the physical features and stability of optical solitons propagation in long-distance communication fibres. Equation (8) is commonly known as the dispersion-managed nonlinear Schrödinger equation (DMNLSE), and it governs the propagation of a dispersion-managed soliton through a polarization preserving optical fibre with damping and periodic amplification [30]. The interaction between linear dispersion and the nonlinear self-pulse modulation effect results from the different velocities at which

different parts of the spectrally modified pulse travel. The dispersion wants to broaden the pulse width, while the nonlinearity intends to sharpen the pulse peak. Suppose the dispersion coefficient k is less than zero. In that case, the pulse continually broadens, while for $k>0$, there may be a balance between dispersion and self-pulse modulation such that the pulse width stays constant. To quantify the interaction between dispersion and self-pulse modulation, we define normalised parameters for the time and the distance. The same procedure as outlined in 3.1 above hold for the conservation of law in the case of variable coefficients.

4 Finite Difference Method

To apply the finite difference method, we divide the domain into N_x sections, each of length

$h = (x_f - x_0)/N_x$ along the x −axis. The time domain is also divided into

N segments each of duration $\tau = T/N$. By expanding function values at grid points in a Taylor series, approximations to the differential equation involving algebraic relations between grid point values can be obtained [12]. For the explicit Euler scheme, we have

$$i \frac{U_i^{n+1} - U_i^n}{\tau} + k \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} + q \ |U_i^n|^2 U_i^n = 0, \tag{9}$$

for $i = 1,2, \dots, N_x - 1$.

$$\frac{U_i^{n+1} - U_i^n}{\tau} - \frac{i\tau k}{2h^2}(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1} + U_{i+1}^n - 2U_i^n + U_{i-1}^n) - \frac{i\tau q}{2}|U_i^n|^2(U_i^{n+1} - U_i^n) = 0, \tag{10}$$

For the Crank-Nicholson scheme, we have

$$\frac{U_i^{n+1} - U_i^n}{\tau} - \frac{i\tau k}{2h^2}(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1} + U_{i+1}^n - 2U_i^n + U_{i-1}^n) - \frac{i\tau q}{2}|U_i^n|^2(U_i^{n+1} - U_i^n) = 0, \tag{11}$$

which simplifies to

$$\begin{aligned} -r_1 U_{i+1}^{n+1} + (1 + 2r_1 - r_2|U_i^n|^2)U_i^{n+1} - r_1 U_{i-1}^{n+1} \\ = r_1 U_{i+1}^n + (1 + 2r_1 - r_2|U_i^n|^2)U_i^n + r_1 U_{i-1}^n, \end{aligned} \tag{12}$$

with $r_1 = \frac{i\tau k}{4h^2}, \quad r_2 = \frac{i\tau q}{2}$.

With the Dirichlet type of boundary condition on x_0 and x_f this can be cast into the following tridiagonal system of equations.

$$
\begin{bmatrix}
1+r & -r_1 & 0 & \cdots & 0 & 0 \\
-r_1 & 1+r & -r_1 & \ddots & 0 & 0 \\
0 & -r_1 & 1+r & \ddots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ddots & 1+r & -r_1 \\
0 & 0 & 0 & \cdots & -r_1 & 1+r
\end{bmatrix}
\begin{bmatrix}
U_1^{n+1} \\
U_2^{n+1} \\
\vdots \\
\vdots \\
U_{N_x-1}^{n+1} \\
U_{N_x}^{n+1}
\end{bmatrix}
$$
$$
=
\begin{bmatrix}
1-r & r_1 & 0 & \cdots & 0 & 0 \\
r_1 & 1-r & r_1 & \ddots & 0 & 0 \\
0 & r_1 & 1-r & \ddots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ddots & 1-r & r_1 \\
0 & 0 & 0 & \cdots & r_1 & 1-r
\end{bmatrix}
\begin{bmatrix}
U_1^n \\
U_2^n \\
\vdots \\
\vdots \\
U_{N_x-1}^n \\
U_{N_x}^n
\end{bmatrix}
+
\begin{bmatrix}
r_1(U_0^{n+1} + U_0^n) \\
0 \\
\vdots \\
\vdots \\
0 \\
r_1(U_{N_x}^{n+1} + U_{N_x}^n)
\end{bmatrix},
$$

where $r = 2r_1 - r_2|U_i^n|^2$.

The system of equations can be solved very efficiently, and its unconditional stability can be obtained.

5 Stability of Finite Difference Method

We shall analyse the stability of the explicit Euler, implicit Euler and the Crank-Nicholson schemes.

The idea here is to check what happens to our solution obtained through this scheme when a small perturbation is introduced to the system. Does it remain a reasonable solution, or there is a ``blow-up``? Indeed, the explicit method is sensitive to small perturbations. Thus, a condition relating to the time step and spatial step size guarantees a solution has to be established.

5.1 Explicit Scheme

We use the explicit scheme with a first-order discretisation concerning time and a second order discretisation with respect to space is as follows:

$$i\frac{U_i^{n+1} - U_i^n}{\tau} + k\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} + q|U_i^n|^2 U_i^n = 0. \tag{13}$$

We write the wave function as the sum of real and imaginary parts. This provides a convenient approach to establishing the stability of the explicit scheme in equation (13):

$$U = \xi + i\eta. \tag{14}$$

Substituting equation (14) into (1), we obtain

$$i\left(\frac{\partial \xi}{\partial t} + i\frac{\partial \eta}{\partial t}\right) + k\left(\frac{\partial^2 \xi}{\partial t^2} + ik\frac{\partial^2 \eta}{\partial t^2}\right) + q|\xi + i\eta|^2(\xi + i\eta) = 0, \tag{15}$$

or

$$i\frac{\partial \xi}{\partial t} - \frac{\partial \eta}{\partial t} + k\frac{\partial^2 \xi}{\partial t^2} + ik\frac{\partial^2 \eta}{\partial t^2} + q(\xi^2 + \eta^2)\xi + iq(\xi^2 + \eta^2)\eta = 0. \tag{16}$$

Separating the real and imaginary parts, we obtain the following system of equations:

$$\frac{\partial \xi}{\partial t} + k\frac{\partial^2 \eta}{\partial t^2} + q(\xi^2 + \eta^2)\eta = 0,$$

$$-\frac{\partial \eta}{\partial t} + k\frac{\partial^2 \xi}{\partial t^2} + q(\xi^2 + \eta^2)\xi = 0. \tag{17}$$

If we apply the explicit scheme (13), we can build an evolution matrix as follows:

$$\frac{\xi_i^{n+1} - \xi_i^n}{\tau} + k\frac{\eta_{i+1}^n - 2\eta_i^n + \eta_{i-1}^n}{h^2} + q\left[(\xi_i^n)^2 + (\eta_i^n)^2\right]\eta_i^n = 0,$$

$$-\frac{\eta_i^{n+1} - \eta_i^n}{\tau} + k\frac{\xi_{i+1}^n - 2\xi_i^n + \xi_{i-1}^n}{h^2} + q[(\xi_i^n)^2 + (\eta_i^n)^2]\xi_i^n = 0, \tag{18}$$

which can be written as

$$\xi_i^{n+1} = \xi_i^n - \frac{k\tau}{h^2}(\eta_{i+1}^n - 2\eta_i^n + \eta_{i-1}^n) - q[(\xi_i^n)^2 + (\eta_i^n)^2]\eta_i^n = 0,$$

$$\eta_i^{n+1} = \eta_i^n + \frac{k\tau}{h^2}(\xi_{i+1}^n - 2\xi_i^n + \xi_{i-1}^n) + q\left[(\xi_i^n)^2 + (\eta_i^n)^2\right]\eta_i^n = 0, \tag{19}$$

$$\begin{pmatrix} \xi_i^{n+1} \\ \eta_i^{n+1} \end{pmatrix} = \begin{pmatrix} T_{11}^n & T_{12}^n \\ T_{21}^n & T_{22}^n \end{pmatrix}\begin{pmatrix} \xi_i^n \\ \eta_i^n \end{pmatrix},$$

where,

$$\widetilde{T_{11}^n} = \widetilde{T_{22}^n} = \delta_{i,r},$$
$$\widetilde{T_{12}^n} = -\widetilde{T_{21}^n} = -\frac{k\tau}{h^2}\left(\delta_{i+1,r} - 2\delta_{i,r} + \delta_{i-1,r}\right) - \tau q \delta_{i,r}\left[\left(\xi_i^n\right)^2 + \left(\eta_i^n\right)^2\right]. \tag{20}$$

This evolution matrix acts in the vector space $R(W)$ of the columns:

$$W^n = \begin{pmatrix} \xi_i^n \\ \eta_i^n \end{pmatrix},$$

$$W^{n+1} = \widetilde{T^{n+1}} W^n = \widetilde{T^{n+1}}\widetilde{T^n} W^{n-1} = \cdots = \prod_{k=n+1}^{1} \widetilde{T^k}\, W^0. \tag{21}$$

Now we will prove stability with respect to small perturbation (because we are considering nonlinear equations) of initial conditions [23,31]]. In other words, we are exploring the boundedness of the discrete solutions in terms of small perturbation of the initial data. It is worth noting that matrix $\widetilde{T}$ is nonlinear and depends on the initial condition. Introducing a small perturbation of the initial condition, we obtain a new matrix dT^k by the evolution of the differential from the matrix $\widetilde{T}$ for $d\xi$ and $d\eta$,

$$dW^{n+1} = \prod_k dT^k\, dW^0, \tag{22}$$

To obtain the stability conditions, we require the boundedness of the operator $\prod_k d\,T^k$ in the sense of the spectral norm. That is, the norm $\left|\left|\prod_k dT^k\right|\right| \le C$ bounded by a constant C. Therefore, the sufficient condition for stability is in the form

$$\left|\left|dT^k\right|\right| \le \exp(a\tau), \tag{23}$$

for some constant a, independent of τ. Moreover, the above condition is also sufficient for stability in case $a = a(\tau, h) \le const < \infty$ with dependence between τ and h such that $\tau = f(h)$ and $\tau, h \to 0$. It is clear from the aforementioned stability condition that the explicit scheme is conditionally stable, and this is further explored as follows.

To calculate the matrix $\widetilde{T}$ we should upper estimate the function $\rho(\tau, h)$ via upper estimation of all matrices using the matrix spectral norm [23]. Thus, we have

$$\left|\left|dT_{11}^n\right|\right|^2 \le 1,$$
$$\left|\left|dT_{12}^n\right|\right|^2 \le \frac{6\tau k}{h^2} + \tau q \max_i \left[\left(\xi_i^n\right)^2 + \left(\eta_i^n\right)^2\right] + 2\tau|q\xi_i^n\eta_i^n|,$$
$$\left|\left|dT_{12}^n\right|\right|^2 \le \frac{6\tau k}{h^2} + \tau q \max_i \left[\left(\xi_i^n\right)^2 + \left(\eta_i^n\right)^2\right] + 2\tau q|\xi_i^n\eta_i^n|, \tag{24}$$
$$\left|\left|dT_{11}^n\right|\right|^2 \le 1.$$

Let us divide the matrix $dT^n = S^n + A^n$ for the symmetric S^n and antisymmetric A^n parts. We use Schwartz and triangle inequalities to estimate mixed terms $\xi_i^n\eta_i^n$ and the commutator $[S^n, A^n]$. It is perceptible that the sub-matrices dT_{ii} are divided into three matrices. The identity matrix E, the symmetric one (but without the identity part $S^{n+1} - E$) and the antisymmetric one A^{n+1} (with nonzero elements for $r = i - 1$ and $r = 1 + 1$) yields

$$\left|\left|dT^{n+1}\right|\right|^2 = \left|\left|dT^{n+1*}dT^{n+1}\right|\right| = \left|\left|(E - S^{n+1} + A^{n+1})(E + S^{n+1} + A^{n+1})\right|\right|$$
$$\le 2\left|\left|S^{n+1} - E\right|\right| + \left|\left|A^{n+1}\right|\right|^2 + \left|\left|[S^{n+1}, A^{n+1}]\right|\right| + \left|\left|S^{n+1} - E\right|\right|^2$$
$$\le 1 + 2\tau q \max_i \left[\left(\xi_i^n\right)^2 + \left(\eta_i^n\right)^2\right] + \frac{6\tau k}{h^2} + 3\tau q \max_i \left[\left(\xi_i^n\right)^2 + \left(\eta_i^n\right)^2\right]^2$$

$$+4\tau \max_i \left[\left(\xi_i^n\right)^2 + \left(\eta_i^n\right)^2\right] \left\{\frac{6\tau k}{h^2} + 3\tau q \max_i \left[\left(\xi_i^n\right)^2 + \left(\eta_i^n\right)^2\right]\right.$$

$$+4\tau^2 q \max_i \left[\left(\xi_i^n\right)^2 + \left(\eta_i^n\right)^2\right]^2$$

And using the discrete form of the conservation law obtained

$$I = \sum_{i=1}^{N} |U_i^n|^2 = \sum_{i=1}^{N} |U_i^0|^2, \tag{25}$$

$$\left|\left|dT^{n+1}\right|\right|^2 \leq 1 + 2\tau MI + 12\frac{\tau^2 k}{h^2} + 36\frac{\tau^2 k^2}{h^4} \leq \exp(\rho\tau), \tag{26}$$

where,

$$M = \max(q),$$
$$\rho = 2\tau MI + 12\frac{\tau k}{h^2} + 36\frac{\tau k^2}{h^4}.$$

Hence, we have obtained a necessary condition for the stability of the scheme in equation (13). The scheme is stable if $\rho \leq$ constant $< \infty$ with the condition that , $h \rightarrow 0$. This is the conditional stability of the scheme. It means that it is required $\tau \rightarrow 0$ faster than $h \rightarrow 0$ or $\tau <$ (constant)h^4.This means that as we decrease the spatial interval h for better accuracy, we must also decrease the time step τ at the cost of more computations in order not to lose the stability.

5.2 Stability in the case of the variable coefficient case

We shall follow the same approach presented in **5.1** to prove the stability of variable coefficients NLS equation (8). First, we note that the conservation law of energy ceases to hold for the case where the dispersion coefficient is a variable. Here, we take k to be constant and present an estimate for the case where the nonlinear coefficient is a function of the spatial coordinate. In this case, our evolution matrix can be built as was done in equations (18) to (20). We shall present an upper estimation of the evolution matrix here.

$$\left|\left|dT_{11}^n\right|\right|^2 \leq 1,$$
$$\left|\left|dT_{12}^n\right|\right|^2 \leq \frac{6\tau k}{h^2} + \tau \max_i q_i \max_i \left[\left(\xi_i^n\right)^2 + \left(\eta_i^n\right)^2\right] + 2\tau \max_i q_i \left|\xi_i^n \eta_i^n\right|,$$
$$\left|\left|dT_{12}^n\right|\right|^2 \leq \frac{6\tau k}{h^2} + \tau \max_i q_i \max_i \left[\left(\xi_i^n\right)^2 + \left(\eta_i^n\right)^2\right] + 2\tau \max_i q_i \left|\xi_i^n \eta_i^n\right|, \tag{27}$$
$$\left|\left|dT_{11}^n\right|\right|^2 \leq 1.$$

$$\left|\left|dT^{n+1}\right|\right|^2 = \left|\left|dT^{n+1*}dT^{n+1}\right|\right| = \left|\left|(E - S^{n+1} + A^{n+1})(E + S^{n+1} + A^{n+1})\right|\right|$$
$$\leq 2\left|\left|S^{n+1} - E\right|\right| + \left|\left|A^{n+1}\right|\right|^2 + \left|\left|[S^{n+1}, A^{n+1}]\right|\right| + \left|\left|S^{n+1} - E\right|\right|^2$$
$$\leq 1 + 2\tau \left\{\max_i q_i \max_i \left[\left(\xi_i^n\right)^2 + \left(\eta_i^n\right)^2\right]\right\}$$
$$+\frac{4\tau k}{h^2} + 3\tau \left[\max_i q_i \max_i \left[\left(\xi_i^n\right)^2 + \left(\eta_i^n\right)^2\right]\right]^2$$
$$+4\tau \left\{\max_i q_i \max_i \left[\left(\xi_i^n\right)^2 + \left(\eta_i^n\right)^2\right]\right\}$$
$$\times \left\{\frac{4\tau k}{h^2} + 3\tau \max_i q_i \max_i \left[\left(\xi_i^n\right)^2 + \left(\eta_i^n\right)^2\right]\right.$$
$$+4\tau^2 \max_i q_i \max_i \left[\left(\xi_i^n\right)^2 + \left(\eta_i^n\right)^2\right]^2.$$

Hence, we obtain a similar stability condition as

$$\left| \left| dT^{n+1} \right| \right|^2 \le 1 + 2\tau MI + 8\frac{\tau^2 d}{h^2} + 16\frac{\tau^2 d^2}{h^4} + 5\tau MI \left[5MI + 8\tau\frac{k}{h^2} \right] \le \exp(\rho\tau), \tag{28}$$

where,

$$M = \max(qI),$$
$$\rho = 2\tau MI + 8\frac{\tau d}{h^2} + 16\frac{\tau d^2}{h^4} + 5\tau MI \left[5MI + 8\tau\frac{k}{h^2} \right].$$

6 Convergence of Finite Difference Schemes Used

6.1 Convergence of the explicit Euler scheme

In this section, we prove that a solution of equation (13) converges to a solution of equation (1) which are differentiable with respect to time and (twice) to x. We substitute a solution of the form

$$W_i^n = E_i^n + R_i^n = \begin{pmatrix} e\xi_i^n + r\xi_i^n \\ e\eta_i^n + r\eta_i^n \end{pmatrix}, \tag{29}$$

where ES is an exact solution and η is the error between a numerical solution and the exact solution of the NLS equation. We obtain the equation for R,

$$\frac{r\xi_i^{n+1} - r\xi_i^n}{\tau} + k\frac{r\xi_{i+1}^n - 2r\xi_i^n + r\xi_{i_1}^n}{h^2} + \frac{e\xi_i^{n+1} - e\xi_i^n}{\tau}$$
$$+k\frac{e\xi_{i+1}^n - 2e\xi_i^n + e\xi_{i_1}^n}{h^2} + q[(r\xi_i^n + e\xi_i^n)^2$$
$$+(r\eta_i^n + e\eta_i^n)^2](r\xi_i^n + e\xi_i^n) = 0.$$

We employ the conservation law previously derived

$$I_e = \sum_{i=1}^{N} \left[\left(r\xi_i^n \right)^2 + \left(e\eta_i^n \right)^2 \right] = \sum_{i=1}^{N} \left[\left(r\xi_i^0 \right)^2 + \left(e\eta_i^0 \right)^2 \right], \tag{30}$$

$$r\xi_i^{n+1} - r\xi_i^n + \tau k\frac{r\xi_{i+1}^n - 2r\xi_i^n + r\xi_{i+1}^n}{h^2} + \tau q \left[\left(e\xi_i^n \right)^2 + \left(e\eta_i^n \right)^2 \right] r\eta_i^n$$
$$- \left[\left(e\xi_i^n \right)^2 + \left(e\eta_i^n \right)^2 \right] (r\eta_i^n + \tau q[(e\xi_i^n)^2 + (e\eta_1^n)^2]e\eta_i^n$$
$$+\tau q \left[\left(e\xi_i^n + r\xi_i^n \right)^2 + \left(e\eta_i^n + r\eta_i^n \right)^2 \right] \left(e\xi_i^n + r\xi_i^n \right)$$
$$= -\tau \left(\frac{e\xi_i^{n+1} - e\xi_i^n}{\tau} - k\frac{e\eta_{i+1}^{n+1} - 2e\eta_i^n + e\eta_{i-1}^n}{h^2} \right)$$
$$+\tau q \left[\left(e\xi_i^n \right)^2 + \left(e\eta_i^n \right)^2 \right] e\eta_i^n.$$

The right-hand side of the equation for the differential solution is of the order $\theta(\tau + h + h^2)$,

while to represent the left-hand side, we use the matrix T. Hence we obtain

$$r\xi_i^{n+1} = T_i^{n+1} R_i^n + \tau q \left[\left(e\xi_i^n \right)^2 + \left(e\eta_i^n \right)^2 \right] \left(e\eta_i^n + r\eta_i^n \right) + \tau q \left[\left(e\xi_i^n + r\xi_i^n \right)^2 \right.$$
$$\left. + \left(e\eta_i^n + r\eta_i^n \right)^2 \right] \left(e\eta_i^n + r\eta_i^n \right) + \theta(\tau + h + h^2).$$

Let us define the norm of the numerical solution as

$$\left\|U^n\right\| = h\left(\sum_i |U_i^n|^2\right)^{1/2}. \tag{31}$$

Now we upper estimate one element of the vector R^n

$$\left\|r_\xi^{n+1}\right\| \le \left\|T^{n+1}\right\| \left\|R^n\right\| + \tau(qI)\left\|\eta^n\right\| - \tau(qI_e)\left\|\eta^n\right\| + \theta(\tau + h + h^2). \tag{32}$$

Next, we write the convergence condition for all R^{n+1} matrix components up to the choice of the initial error $\left\|R^0\right\| = 0$,

$$\left\|R^{n+1}\right\| \le Q + \theta(\tau + h + h^2), \tag{33}$$

where θ is defined as

$$Q = 4|\max(q)\, I^{3/2} - \max(q)\, I_e^{3/2}|. \tag{34}$$

6.2 Crank-Nicholson scheme

Convergence for Crank-Nicholson Scheme. We start from the discretized form as given below:

$$\frac{U_i^{n+1}-U_i^n}{\tau} + \frac{k}{2}\left(\frac{U_{i+1}^{n+1}-2U_i^{n+1}+U_{i-1}^{n+1}}{h^2}\frac{U_{i+1}^n-2U_i^n+U_{i-1}^n}{h^2}\right) + q|U_i^{n+1}|^2\left(\frac{U_i^{n+1}+U_i^n}{2}\right) = 0. \tag{35}$$

Now we let $r_x = \frac{\tau k}{2h^2}$, we have the above equation as

$$\begin{aligned} &U_{i+1}^{n+1} + r_x(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}) = -r_x(U_{i+1}^n - 2U_i^n + U_{i-1}^n)\\ &-\tau q|U_i^n|^2\left(\frac{U_i^{n+1}+U_i^n}{2}\right), \end{aligned} \tag{36}$$

where τ and h are the time and space steps, respectively and $U_i^n = U(n\tau, ih)$. The scheme is consistent with the NLS equation, and the local truncation error is $O(\tau^2 + h^2)$. At each discrete time level, only a set of linear algebraic equations has to be solved to obtain the value of the wave function U_i^{n+1} at the following time $n + 1$. Similar to the explicit scheme discussed earlier, the Crank-Nicholson scheme also has constant energy and charge, which are analogues of conservation law

$$\begin{aligned} E &= \sum_i \frac{k}{2}\left(\left|\frac{U_{i+1}^{n+1} - U_i^{n+1}}{h}\right|^2 + \left|\frac{U_{i+1}^n - U_i^n}{h}\right|^2\right) - \frac{q}{2}\sum_i h\,|U_i^{n+1}|^2|U_i^n|^2,\\ Q^n &= \sum_i \frac{h}{2}(|U_i^n|^2 + |U_i^{n+1}|^2). \end{aligned} \tag{37}$$

We now briefly establish the proof of the convergence of the Crank-Nicholson scheme as follows.
Assume that $U^n(x) = W^n(x) + w^n(x), \quad x = ih$, where $W^n(x)$ is the exact solution, U^n is the numerical solution and w^n is the error. From the conservation law, we have

$$\left\|U^n\right\|^2 \le \left\|W^0\right\|^2 + \left\|W^1\right\|^2, \tag{38}$$

With

$$\left\|U^n\right\| = \langle U^n, U^n\rangle = h\sum_i \overline{U_i^n}U_i^n.$$

Therefore, the error can be estimated as follows

$$\iota \frac{w_i^{n+1} - w_i^n}{\tau} = -\frac{1}{2h^2}\left(w_{i+1}^{n+1} - 2w_i^{n+1} + w_{i-1}^{n+1} + w_{i+1}^n - 2w_i^n + w_{i-1}^n\right) - \frac{q}{2}\left|\overline{U_i^n}\right|^2\left(w_i^{n+1} + w_i^n\right) + G_i^n + F_i^n, \tag{39}$$

where F_i^n is the truncation error which is of order $O(\tau^2 + h^2)$ and

$$G_i^n = \frac{q}{2}(W_i^{n+1} + W_i^n)(|W_i^n|^2 - |U_i^n|^2) + \frac{q}{2}(w_i^{n+1} + w_i^n)(|w_i^n|^2 - |w_i^n|^2). \tag{40}$$

Then we obtain

$$\left||G^n|\right| \leq M_1\left||w^n|\right| + M_2\left(\left||w^{n+1}|\right| + \left||w^n|\right|\right), \tag{41}$$

where M_1 and M_2 are two constants only depend on the initial data U_i^0 and U_i^1.

Multiplying equation (39) by $\left(\overline{w^{n+1}} + \overline{w^n}\right)$,

$$\left[\iota \frac{w_i^{n+1} - w_i^n}{2\tau} = -\frac{k}{2h^2}(w_{i+1}^{n+1} - 2w_i^{n+1} + w_{i-1}^{n+1} + w_{i+1}^n - 2w_i^n + w_{i-1}^n) \right. $$
$$\left. -\frac{q}{2}|\overline{U_i^n}|\left(\overline{U_i^{n+1}} + w_i^n\right) + G_i^n + F_i^n\right]\left(\overline{w^{n+1}} + \overline{w^n}\right) \tag{42}$$

$$\frac{\left||w_i^{n+1}|\right|^2 - \left||w_i^n|\right|^2 + w_i^{n+1}\overline{w_i^n} - w_i^n\overline{w_i^{n+1}}}{\tau} = -\frac{k}{2h^2}\left(w_{i+1}^{n+1}\overline{w_i^{n+1}} - 2\left||w_i^{n+1}|\right|^2\right)$$
$$+ w_{i-1}^{n+1}\overline{w_i^{n+1}} + w_{i+1}^n\overline{w_i^{n+1}} - 2w_i^n w_i^{n+1} + w_{i+1}^n\overline{w_i^{n+1}} + w_{i+1}^{n+1}\overline{w_i^n}$$

$$- 2w_i^{n+1}\overline{w_i^n} + w_{i-1}^{n+1}\overline{w_i^n} + w_{i-1}^{n+1}\overline{w_i^n} - 2\left||w_i^n|\right|^2 + w_{i-1}^n\overline{w_i^n}\right) \tag{43}$$

$$- \frac{q}{2}|w_i^n|^2\left[\left||w_i^{n+1}|\right|^2 + \left||w_i^n|\right|^2 + w_i^{n+1}\overline{w_i^n} + w_i^n\overline{w_i^{n+1}}\right]$$

$$+ \left\langle G^n, \left(\overline{w_i^{n+1}} + \overline{w_i^n}\right)\right\rangle + \left\langle F^n, \left(\overline{w_i^{n+1}} + \overline{w_i^n}\right)\right\rangle.$$

summing over i and taking the imaginary part, we get

$$\frac{\left||w_i^{n+1}|\right|^2 - \left||w_i^n|\right|^2}{\tau} = Im\left\langle G^n, \left(\overline{w_i^{n+1}} + \overline{w_i^n}\right)\right\rangle + \left\langle F^n, \left(\overline{w_i^{n+1}} + \overline{w_i^n}\right)\right\rangle. \tag{44}$$

From equations (39), (44), we arrive at

$$\left||w_i^{n+1}|\right|^2 = \left||w^0|\right|^2 + \left||w^1|\right|^2 + \tau C\sum_{m=1}^{n+1}\left||w^m|\right|^2 + \tau\sum_{m=1}^{n}\left||F^m|\right|^2, \tag{45}$$

where C is a constant depending on the initial condition. We assume that τ is small such that $1 - C\tau \equiv \Delta > 0$; then we obtain the result

$$\left||w^{n+1}|\right|^2 \leq \frac{1}{\Delta}\left(\left||w^0|\right|^2 + \left||w^1|\right|^2 + \tau\sum_{m=1}^{n}\left||F^m|\right|^2\right)exp\left(\frac{C(n+1)\tau}{\Delta}\right). \tag{46}$$

This means that the error is bounded by the initial and truncation errors, so the scheme is convergent.

7 Conclusions

In this paper, we first started by introducing solitary waves and showed their variety of applications in many areas of nonlinear science. Then we introduced the CNSE, which is our model for this work. First, we proved conservation laws on this system and stability. We then reduced the CNSE to the Manakov system, which has the known analytical solution; we could compare the finite difference schemes and observe how good this method can approximate solutions. Here, we found the implicit scheme to converge to the analytical solution faster than expected. As an improved Manakov system, we considered the case of CNSE with constant coefficients, in which interactions of waves were also observed [32] in Bose-Einstein condensate. It is here where we see the interactions of the two waves. Finally, we considered the case of variable coefficients, which resulted in the dispersion management of solitary waves [33].

We draw from this conclusion that solitons or solitary waves can be constructed depending on the parameters for both dispersion and nonlinear terms in CNSE. It remains a significant challenge to find the best parameters to result with solitons, especially in the case of variable coefficient, where there is a balance between the dispersion and nonlinear coefficients. In [29], they are working on solitons. They are finding N-soliton solutions. Also, they investigate different parameters suitable to produce solitons. More complicated nonlinear equations were subsequently dispatched when Ablowitz, Kaup, Newell, and Segur showed how to make the method of solving them more systematic with a procedure now known as the AKNS method [34]. They can find parameters that can give reasonable soliton solutions depending on equations and parameters.

The main present and future problem is finding a balance between the dispersive coefficient and nonlinearity to maintain a soliton solution. This is because the propagation speed of a wave is frequency-dependent, the transmitted pulses tend to spread (an effect called dispersive spreading), and the signal, in turn, tends to break up. Since the inception of optical fibres, researchers have been looking for ways to combat dispersive spreading because of the severe limits it imposes on the capacity of optical communication systems [34]. Pulses that spread to a large extent will overlap, making it increasingly difficult to separate them from one another; the result is the degradation of the signal. The mathematical theory of nonlinear wave equations with rapidly varying coefficients is still not fully resolved and is a subject of ongoing research worldwide. A primary goal is to understand the pulse behaviour when the variations are substantial; this would provide a foundation for explaining several dispersion-management experiments that have been performed.

Competing Interests

Authors have declared that no competing interests exist.

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