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Characterization of Orthogonal Projectors

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Abstract

Let H be a Hilbert space and M be a closed linear subspace of H. Then by projection theorem $H = M \oplus M^{\perp}$. This theorem suggests that the result has something to do about a notion in Hilbert spaces which is analogous to and a generalization of the familiar idea of Orthogonal or perpendicular projection of a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ upon a linear subspace of $\mathbb{R}^2$ or $\mathbb{R}^3$ respectively. In this paper we give a complete operator characterization of orthogonal projections.Specifically we show that P is an orthogonal projector onto $\Re_P = M$ if and only if P is self-adjoint and idempotent. We also consider the algebraic formulation of invariance, reduction, orthocomplementation and orthogonality.

Keywords: Orthogonal projector; self-adjoint; idempotent; invariance; reduction.

1 Introduction

Let H be a Hilbert space and M be a closed Linear Subspace of H. Then (by projection theorem [1])

$$H = M \oplus M^{\perp}$$

Then for any $x \in H$, there are unique $x' \in M$ and $x'' \in M^{\perp}$ such that

$$x = x\prime + x\prime\prime$$

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Similarly if $y \in H$, we can write $y = y' + y''$ for unique $y' \in M$ and $y'' \in M^{\perp}$.

Definition 1. Let H be a Hilbert space and M be a closed linear subspace of H. For each $x \in H$; consider the unique decomposition $x = x' + y'$, where $x' \in M$ and $y' \in M^{\perp}$. This decomposition is guaranteed by the Projection theorem [1]. The component x' is called the **orthogonal projection** of the vector $x \in M$. (Likewise, y' is the orthogonal projection of the vector x on $M^{\perp}$). Note that if x is in M, then $x' = x$ and $y' = 0$. The mapping $P_M : H \longrightarrow M$ defined by

$$P_M x = x'$$

(with x, x' as just described) is called the **orthogonal projection operator onto** M or the **orthoprojector onto** M or the **orthogonal projector onto** M. (Note that the range of P_M is M).

A mapping $P : H \longrightarrow H$ is called an **orthogonal projector** or **orthoprojector** on H if there is a closed linear subspace M of H such that $P = P_M$, that is, P equals the orthoprojector onto M.

Remark 1. *If $x \in M$, then $x = x + \overline{0}$ is the unique decomposition of x in $M \oplus M^{\perp}$. Thus $x \in M$ implies*

$$P_1 x = x, P_2 x = \overline{0},$$

where P_1, P_2 denote the orthoprojectors onto the closed linear subspaces $M, M^{\perp}$, respectively. Thus $P_1|_M$ is the identity map on M and $P_2|_{M^{\perp}}$ is the zero map.

Remark 2. *It is obvious that for each $x \in H, P_1 x$ is the unique element of the closed linear subspace M whose distance from x equals dist (x, M), that is*

$$\|x - P_1 x\| = \mathrm{dist}(x, M) \text{ [2]}$$

Remark 3. *For each closed linear subspace M of H, we have $P_1 + P_2 = I$, where P_1, P_2 are the orthoprojectors onto $M, M^{\perp}$ respectively, I being an identity map on H. Thus $P_2 = I - P_1$. Thus if P is the orthoprojector onto the closed linear subspace M of H, then $I - P$ is the orthoprojector onto the orthogonal complement $M^{\perp}$ of M with respect to H.*

We are naturally curious with many questions about an orthoprojector P : Is P linear? bounded? What properties characterize an orthoprojector? Proposition 1 is decisive in even answering more. In this paper most definitions can be found in [3], [4],[5],[6],[7], [8], [9].

2 Properties of Orthoprojectors

Proposition 1. *Let H be a Hilbert space, M be a closed linear subspace of H and let P and Q be the orthoprojectors onto M and $M^{\perp}$ respectively. Then*
(i) $P : H \to H$ is linear.
(ii) Both P and Q are bounded and $\|P\| \leqslant 1, \|Q\| \leqslant 1$.
(iii) $\mathfrak{R}_P = M$
(iv) $\eta_P = M^{\perp}$
(v) $\eta_P \perp \mathfrak{R}_P$ and $\eta_Q \perp \mathfrak{R}_Q$
(vi) $I = P + Q$
(vii) P, Q are self-adjoint.
(viii) P, Q are idempotent i.e $P^2 = P, Q^2 = Q$

Proof. (i) Let $\alpha, \beta \in \mathbb{K}$ and consider the element $\alpha x + \beta y \in H$.

$$\begin{aligned} \alpha x + \beta y &= \alpha (x' + x'') + \beta (y' + y'') \\ &= (\alpha x' + \beta y') + (\alpha x'' + \beta y'') \end{aligned}$$

Since $M, M^{\perp}$ are linear subspaces so $\alpha x' + \beta y' \in M$ for $x', y' \in M$ and $\alpha x'' + \beta y'' \in M^{\perp}$ for $x'', y'' \in M^{\perp}$.Thus $\alpha x + \beta y$ has a decomposition $(\alpha x' + \beta y') + (\alpha x'' + \beta y'')$ in $M \oplus M^{\perp}$ and since this decomposition is unique $P(\alpha x + \beta y) = \alpha x' + \beta y'$. But $Px = x\prime$, $Py = y\prime$. Hence $P(\alpha x + \beta y) = \alpha x' + \beta y' = \alpha Px + \beta Py$ which implies P is linear . Similarly, if we define $Q : H \to H$ by $Qz = z''$ $\forall z \in H$ (where $z = z' + z''$ is the decomposition of z with respect to the direct sum $M \oplus M^{\perp}$ where $z' \in M, z'' \in M^{\perp}$) then Q is also linear.
(ii) Let $x \in H$ and $x = x' + x''$ be the decomposition of x with $x' \in M$ and $x'' \in M^{\perp}$. Then $Px = x'$. So $\|Px\| = \|x'\|$ i.e $\|Px\|^2 = \|x'\|^2$ Therefore;

$$\|Px\|^2 \leqslant \|x'\|^2 + \|x''\|^2$$

Since $x' \in M$ and $x'' \in M^{\perp}$, we have $x' \perp x''$ and thus (from $x = x' + x''$) we have by Pythagorean theorem [10]

$$\|x\|^2 = \|x'\|^2 + \|x''\|^2$$

Hence $\|Px\|^2 \leq \|x\|^2$ *i.e* $\|Px\| \leq \|x\| \; \forall \; x \in H$ i.e $\|Px\| \leq 1\|x\| \; \forall \; x \in H$ which implies $P \in B(H)$ and $\|P\| \leq 1$.

If $M \neq \{\bar{0}\}$, then there exists $x \in M$ such that $x \neq \bar{0}$. The decomposition for this $x \in H$ is $M \oplus M^{\perp}$ (which is unique) is obviously; $x = x + \overline{0}$ therefore $Px = x$. Thus $Px = x$ for all $x \in M$. Now $\|P\| = \sup\{\|Pz\| : z \in H \text{ and } \|z\| = 1\}$

In particular put $z = \frac{x}{\|x\|}$ $(x \in M$ and $x \neq \overline{0})$. Then

$$\|P\| \geq \left\| P\left(\frac{x}{\|x\|}\right) \right\| = \frac{1}{\|x\|} \|Px\| = \frac{1}{\|x\|}\|x\| = 1$$

Therefore $\|P\| = 1$ if $M \neq \{\bar{0}\}$.Similarly $\|Q\| \leq 1$ and $\|Q\| = 1$ if $M^{\perp} \neq \{\bar{0}\}$.

(iii) For if $z \in H$ then $Pz = z' \in M$, so $Pz \in M \; \forall z \in H$, therefore $P(H) \subseteq M$ i.e

$$\mathfrak{R}_P \subseteq M. \tag{1}$$

If $z \in M$, then we saw that $Pz = z$. So; $P(M) = M$. Therefore

$$P(H) \supseteq P(M) = M \text{ i.e } P(H) \supseteq M \text{ i.e } \mathfrak{R}_P \supseteq M. \tag{2}$$

From (1) and (2)

$$\mathfrak{R}_P = (P(H)) = M.$$

Thus $P : H \to H$ is onto M. Similarly $\|Q\| \leqslant 1$ and $\|Q\| = 1$ if $M^{\perp} \neq \{\overline{0}\}$ and $\mathfrak{R}_Q = M^{\perp}$.

(iv) For if $z \in M^{\perp}$ then z has the unique decomposition $z = \overline{0} + z$ where $\overline{0} \in M$ and $z \in M^{\perp}$. So $Pz = \overline{0}$, i.e $z \in \eta_P$ therefore $M^{\perp} \subseteq \eta_P$. On the other hand, let $x \in \eta_P$ then $Px = \overline{0} \in M^{\perp}$, so $\eta_P \subseteq M^{\perp}$. Thus $\eta_P = M^{\perp}$. Similarly $\eta_Q = M$.

(v) for $M^{\perp} \perp M$ and $M \perp M^{\perp}$, so $\eta_P \perp \mathfrak{R}_P$ and $\eta_Q \perp \mathfrak{R}_Q$.

$$H = \eta_P \oplus \mathfrak{R}_P = \eta_Q + \mathfrak{R}_Q$$

(vi) For any $z \in H$

$$Iz = z = z' + z'' = Pz + Qz = (P + Q)z.$$

Therefore

$$I = (P + Q).$$

So

$$Q = I - P, P = I - Q$$

(vii) For let $x, y \in H$ and $x = x' + x'', y = y' + y''$ be the decompositions of x, y respectively along $M \oplus M^{\perp}$. Thus $Px = x'$, $Py = y'$. Now Since $P \in B(H)$ (So P^* exists [6])

$$\langle Px, y \rangle = \langle x, P^* y \rangle . \tag{3}$$

But

$$\langle Px, y \rangle = \langle x', y \rangle = \langle x', y' + y'' \rangle = \langle x', y' \rangle + \langle x', y'' \rangle .$$

But

$$x' \in M \text{ and } y'' \in M^{\perp} \text{ so } \langle x', y'' \rangle = 0.$$

Thus

$$\langle Px, y \rangle = \langle x', y' \rangle = \langle x', Py \rangle$$

$$= \langle x', Py \rangle + \langle x'', Py \rangle \text{ (for } \langle x'', Py \rangle = 0 \text{ since } x'' \in M^{\perp} \text{ and } Py \in M)$$

$$= \langle x' + x'', Py \rangle = \langle x, Py \rangle \tag{4}$$

from (3) and (4) we obtain

$$\langle x, P^* y \rangle = \langle x, Py \rangle \quad \forall x, y \in H.$$

i.e $P^* = P \ \forall y \in H$. (Indeed $\langle x, P^* y \rangle - \langle x, Py \rangle = 0 \quad \forall x, y \in H$ i.e $\langle x, P^* y - Py \rangle = 0 \ \forall x, y \in H$. Put $x = P^* y - Py$ in particular, therefore $\|P^* y - Py\| = 0 \quad \forall y \in H$ i.e$P^* y - Py = \bar{0} \quad \forall y \in H$ i.e $P^* = P$). Thus P is self-adjoint. Likewise Q is self-adjoint.Indeed;

$$Q = I - P$$

therefore

$$Q^* = (I - P)^* = I^* - P^* = I - P = Q.$$

(viii) Let $x \in H$. Then $Px \in M$. We saw that if $y \in M$ then $Py = y$. Since $Px \in M$ so $P(Px) = Px$ i.e $P^2 x = Px \ \forall x \in H$,therefore $P^2 = P$.Likewise $Q^2 = Q$

□

In general $P \in B(H)$ is an **orthogonal projector** if there is a closed linear subspace of H such that $Px = x' \ \forall x \in H$ where x' is the component of x in the decomposition $x = x' + x''$ along the direct sum $M \oplus M^{\perp}$.More generally, we define a linear operator $P \in B(H)$ to be an orthogonal projection if $P = P^*$ that is P is self-adjoint and P is idempotent i.e $P^2 = P$.

Remark 4. *We usually use the symbol P_M in place of P if we want to exhibit the linear subspace onto which P maps.*

Remark 5. *We have seen that $\|P\| \leq 1$ and $\|P\| = 1$ if $M \neq \{\bar{0}\}$.*
If $M = \{\bar{0}\}$ then $x \in H \Rightarrow x = \bar{0} + x \left(\bar{0} \in M, \bar{0} \in M^{\perp}\right)$. So $Px = \bar{0} \quad \forall x \in H$

$$\|Px\| = \|\bar{0}\| = 0 \quad \forall x \in H$$

therefore

$$\|P\| = 0.$$

Thus if P is an orthogonal projector on H either $\|P\| = 1$ or $\|P\| = 0$.

Remark 6. *Let M be a closed linear subspace of a Hilbert space H and P_M represent the orthogonal projection of H onto M. If $x \in H$, there exist a unique $x_0 \in M$ such that $dist(x, M) = \|x - x_0\|$ and $x - x_0 \perp M$ i.e $x - x_0 \in M^{\perp}$ [3].*

Now $x_0 = P_M x$ (for $x = x_0 + (x - x_0)$ is the decomposition of x in $M \oplus M^{\perp}$ therefore $P_M x = x_0$) Thus,

$$dist(x, M) = \|x - P_M x\| \ \forall \ x \in H.$$

The question which arises is; if $P \in B(H)$, and P is idempotent and self-adjoint, what can we say about P? Proposition 2 answers this question.

Proposition 2. *Let H be a Hilbert space and $P \in B(H)$. The following statements are equivalent.*

(i) P is an orthogonal projector.
(ii) P is idempotent and self-adjoint i.e $P^2 = P$ and $P^* = P$

Proof. (i) $\Rightarrow (ii)$ proved in proposition 1 (vii) and (viii)
Conversely $(ii) \Rightarrow (i)$
P being a linear transformation it follows that $\mathfrak{R}_P$ and η_P are both linear subspaces of H [4]. Since $P \in B(H), \eta_P$ is a closed linear subspace of H. We see that $\mathfrak{R}_P$ is closed. This follows essentially from the idempotency of P. Indeed let $y \in \bar{\mathfrak{R}}_P$, hence there exists a sequence (y_n) of elements of $\mathfrak{R}_P$ such that $y_n \xrightarrow{s} y$. If $x \in \mathfrak{R}_P$ then $Px = x$. Indeed since $x \in \mathfrak{R}_P$ so $x = Py$ for some $y \in H$. Consequently

$$Px = P(Py) = P^2 y = Py$$

(The last equality in the chain follows from idempotency of P). But $Py = x$ so we get $Px = x$ i.e $x \in \mathfrak{R}_P$ implies $Px = x$.

Since $y_n \xrightarrow{s} y$ and $P \in B(H)$ we get

$$Py_n \xrightarrow{s} Py.$$

But $Py_n = y_n$ (since $y_n \in \mathfrak{R}_P$). Thus $y_n \xrightarrow{s} Py$. Also $y_n \xrightarrow{s} y$.

By uniqueness of the strong limit, we get $Py = y$. In other words $y \in \mathfrak{R}_P$.Thus $\bar{\mathfrak{R}}_P \subseteq \mathfrak{R}_P$, i.e $\mathfrak{R}_P$ is closed.

Conclusion: $\mathfrak{R}_P$ is a closed linear subspace of H.

Hence by the projection theorem [1] in Hilbert spaces $H = \mathfrak{R}_P \oplus \mathfrak{R}_P^{\perp}$. Hence if $x \in H$, then $x = x' + x''$, where $x' \in \mathfrak{R}_P$ and $x'' \in \mathfrak{R}_P^{\perp}$.Therefore $\quad Px = Px' + Px''$. Since $x' \in \mathfrak{R}_P, so \quad Px' = x'$. We shall show that $Px'' = \overline{0}$. Since $x'' \in \mathfrak{R}_P^{\perp}, x'' \perp \mathfrak{R}_P$ i.e $\langle x'', Py \rangle = 0 \quad \forall y \in H$. Since P is self adjoint,

$$\left\langle x'', Py \right\rangle = \left\langle P^* x'', y \right\rangle = \left\langle Px'', y \right\rangle.$$

Therefore

$$\left\langle Px'', y \right\rangle = 0 \quad \forall y \in H.$$

Thus

$$Px'' \perp H \quad i \cdot e \quad Px'' = 0.$$

Therefore

$$Px = Px' = x' \in \mathfrak{R}_P.$$

Hence P maps H onto the closed linear subspace $M = \mathfrak{R}_P$ and the component of x in $\mathfrak{R}_P^{\perp}$ belongs to the null space of P. Hence P is the orthogonal projector on H onto the closed linear subspace $\mathfrak{R}_P$. □

Thus the two properties in proposition 2 (ii) are together equivalent to (i) and we obtain a complete operator characterization of an orthogonal projector:

$P \in B(H), P^2 = P$ and P is self-adjoint $\Longleftrightarrow P$ is an orthogonal projector onto $\mathfrak{R}_P$.

We establish next another equivalence.

Example 1. *Let P be in $B(H)$. Then P is an orthoprojector if and only if P is idempotent and $R_P \perp R_{I-P}$.*

SOLUTION. If P is an orthoprojector, then we have already seen in Proposition 1 that P is idempotent and the property $\mathfrak{R}_P \perp \mathfrak{R}_{I-P}$ was also seen. Hence we show that the converse holds. Let $\mathfrak{R}_P = M$. Since P in $B(H)$ is idempotent, M is a closed linear subspace of H, as already seen above. Let $\mathfrak{R}_{I-P} = N$. Now $P^2 = P$ implies $(I-P)^2 = I - 2P + P^2 = I - P$. Since $I - P \in B(H)$ and it is idempotent, it follows that N is a closed linear subspace of H. By hypothesis, $M \perp N$. Moreover, each $x \in H$ can be written as $x = Px + (I-P)x$. Note that Px is in $\mathfrak{R}_P = M, (I-P)x$ is in $\mathfrak{R}_{I-P} = N$. So $H = M + N$, with $M \perp N$. Now $M \perp N$ implies $M \cap N = \{\bar{0}\}$. Thus $H = M \oplus N$, with $M \perp N$. It is thus evident that $N = M^{\perp}$. P is thus an orthoprojector.

Remark 7. *We observe that there is a natural one-to-one correspondence between the set of all closed linear subspaces of a Hilbert space H and the set of all orthoprojectors on H. In view of this, it is possible to express all geometric notions connected with closed linear subspaces in terms of algebraic properties of the orthoprojectors onto these linear spaces. We consider below the algebraic formulation of invariance, reduction, orthocomplementation and orthogonality.*

We now introduce the notion of invariant and reducing linear subspaces for a $T \in B(H)$.

Definition 2. Let H be a Hilbert space, $T \in B(H)$ and M be a closed linear subspace of H. We say that M is **invariant** with respect to T or T **- invariant** if $x \in M$ implies $Tx \in M$. If T is defined on $\mathfrak{D}_T$ (subspace of H) then T is said to be T**- invariant** if $Tx \in M$ for all $x \in M \cap \mathfrak{D}_T$

Trivial cases:

If $M = \{\bar{0}\}$ or $M = H$, then M is always T-invariant. For $x \in H$ implies $Tx \in H$ and $x = \bar{0} = T\bar{0} = \bar{0} \in \{\bar{0}\}$. These are called the **improper T-invariant subspaces**.

η_T is T-invariant for if $x \in \eta_T, Tx = \bar{0} \in \eta_T$.

A question which arises is: If $T \in B(H)$ has an invariant subspace M, what can we say about the adjoint of T i.e T^* ?

Proposition 3. *Let H be a Hilbert space, $T \in B(H)$. Then a closed linear subspace M of H is T-invariant if and only if $M^{\perp}$ is T^*-invariant.*

Proof. Let M be T-invariant. To show that $M^{\perp}$ is T^*-invariant. Let $x \in M^{\perp}$ i.e $\langle x, Ty\rangle = 0 \ \forall \ y \in M$.($M$ is T- invariant $\Longleftrightarrow Ty \in M$ for all $y \in M$) But
$\langle x, Ty\rangle = \langle T^*x, y\rangle$ so $\langle T^*x, y\rangle = 0 \ \forall y \in M$
i.e $T^*x \perp M$ i.e $T^*x \in M^{\perp}$.Conclusion; $x \in M^{\perp}$ implies $T^*x \in M^{\perp}$.Thus $M^{\perp}$ is T^*-invariant.

Thus M is T -invariant implies $M^{\perp}$ is T^*-invariant.By the same result it follows that: $M^{\perp}$ is T^*-invariant, implies $(M^{\perp})^{\perp}$ is $(T^*)^*$-invariant (since $T^* \in B(H)$ and $M^{\perp}$ is a closed linear subspace). But $\left(M^{\perp}\right)^{\perp} = M$ since H is a Hilbert space and $(T^*)^* = T$ for $T \in B(H)$. So $M^{\perp}$ is T^*-invariant implies M is T-invariant.Thus for $T \in B(H)$, if M is a closed linear subspace of H:

M is T-invariant $\Longleftrightarrow M^{\perp}$ is T^*-invariant. □

Remark 8. *T-invariance of M is essentially a geometric concept involving a linear subspace M and its image. This geometric concept of invariance can be translated into a purely algebraic concept involving operators with perfect equivalence as seen in proposition 4;*

Proposition 4. *Let H be a Hilbert space and M be a closed linear subspace of H. Let $T \in B(H) \cdot M$ is then T -invariant if and only if $PTP = TP$ where P is orthogonal projector on H onto M.*

Proof. Let M be T-invariant, so $x \in M$ implies $Tx \in M$. Let $y \in H$. Since P is the orthogonal projector on H onto M, so $Py \in M$. Since M is T-invariant and $Py \in M$, so $TPy = T(Py) \in M$. But $TPy \in M$ implies $P(TPy) = TPy$. Thus $PTPy = TPy \; \forall \; y \in H$,therefore $PTP = TP$.

Conversely let $PTP = TP$.To show that M is T-invariant. Let $x \in M$ we must show that $Tx \in M$. Since $x \in M$. $Px = x$. So $TPx = Tx$. But $TP = PTP$,therefore $TPx = PTPx$. Therefore

$$PTPx = Tx$$

i.e $P(TPx) = Tx$. But $P(TPx) \in \mathfrak{R}_P = M$,therefore $Tx \in M$ i.e $x \in M$ implies $Tx \in M$, i.e M is T-invariant.

□

Definition 3. Let H be a Hilbert space and $T \in B(H)$. A closed linear subspace M of H is said to **reduce** T, if both M and $M^\perp$ are$T$-invariant. In case domain of T is $\mathfrak{D}_T$ then we say that M reduces T if

$$\mathfrak{D}_T = (\mathfrak{D}_T \cap M) + \left(\mathfrak{D}_T \cap M^\perp\right)$$

and $M, M^\perp$ are both T-invariant i.e, $T\left(M\mathfrak{D}_T\right) \subseteq M$ and $T\left(M^\perp \cap \mathfrak{D}_T\right) \subseteq M^\perp$.

Clearly both $\{\overline{0}\}$ and H reduce T and are called **improper reducing subspaces** of T. All other closed linear subspaces M of H which reduce T (i.e $M \neq \{\overline{0}\}$, H) are called **proper reducing subspaces** of H. The operator T is said to be **irreducible** if T has no proper reducing subspaces.

Remark 9. *If $T \in B(H)$ has a proper reducing subspace M. Now $H = M \oplus M^\perp$ by projection theorem [1]. Since T maps M into M and $M^\perp$ into $M^\perp$ hence we can split T into two bounded linear operators; $T|_M, T|_{M^\perp}$ and study these instead of T. Also $T = T|_M + T|_{M^\perp}$.*

Note: $T|_M \in B(M, M)$ *and* $T|_M : M \longrightarrow M$

$T\mid_{M^\perp} \in B\left(M^\perp, M^\perp\right)$ and $T\mid_{M^\perp}: M^\perp \to M^\perp$

It is possible that even $T|_M$, $T|_{M^\perp}$ have themselves reducing subspaces and so on, so that these operators can be further split.

Proposition 5. *Let $T \in B(H)$ be self-adjoint and M be a closed linear subspace of H. Then M reduces T if and only if M is T-invariant.*

Proof. M reduces $T \Longleftrightarrow M, M^\perp$ are both T-invariant.
$M^\perp$ is T-invariant $\Longleftrightarrow$ $(M^\perp)^\perp$ is T^* - invariant $\Longleftrightarrow$ M is T-invariant ($T^* = T$ since T is self adjoint). Thus
M reduces $T \Longleftrightarrow M$ is T-invariant. □

Proposition 6. *Let H be a Hilbert space and $T \in B(H)$. Let M be a closed linear subspace of H and P be the orthogonal projector on H onto M. Then the following statements are equivalent.*
i) M reduces T
iii) M reduces T^*
(ii) $P \longleftrightarrow T$
iv) $P \longleftrightarrow T^*$
v) $M^\perp$ reduces T
vi) $M^\perp$ reduces T^*

Proof. (i) $\Longleftrightarrow$ (iii). Since M reduces T, both M and $M^{\perp}$ are T -invariant. Since P is the orthogonal projector on H onto M then $I-P$ is the orthogonal projector on H onto $M^{\perp}$.

M is T-invariant $\Longleftrightarrow PTP = TP$.
$M^{\perp}$ is T-invariant $\Longleftrightarrow (I-P)T(I-P) = T(I-P)$

$$\Longleftrightarrow T - TP - PT + PTP = T - TP$$
$$\Longleftrightarrow PT = PTP.$$

Thus $PTP = TP$ and $PT = PTP$. Hence $TP = PT$ i.e $P \longleftrightarrow T$.

(iii)$\Longleftrightarrow$ (iv)

$$TP = PT \Longleftrightarrow (TP)^* = (PT)^*$$
$$\Longleftrightarrow P^*T^* = T^*P^* \Longleftrightarrow PT^* = T^*P \text{ (Since P is self- adjoint)}.$$
$$\Longleftrightarrow P \longleftrightarrow T^*$$

(i) $\Longleftrightarrow$ (ii)
M reduces $T \Longleftrightarrow M, M^{\perp}$ are T - invariant.

$$\Longleftrightarrow M^{\perp}, \left(M^{\perp}\right)^{\perp} \text{ are } T^* - \text{invariant.}$$
$$\Longleftrightarrow M^{\perp}, M \text{ are } T^* \text{ invariant.}$$
$$\Longleftrightarrow M \text{ reduces } T^*$$

(i) $\Longleftrightarrow$ (v)
M reduces $T \Longleftrightarrow M, M^{\perp}$ are invariant under T.

$$\Longleftrightarrow M^{\perp}, \left(M^{\perp}\right)^{\perp} \text{ are invariant under } T$$
$$\Longleftrightarrow M^{\perp} \text{reduces} T$$

(v) $\Longleftrightarrow$ (vi)
This holds since (i) $\Longleftrightarrow$ (ii)
Since M reduces $T \Longleftrightarrow M$ reduces T^*. So $M^{\perp}$ reduces $T \Longleftrightarrow M^{\perp}$ reduces T^*.
Finally we show that (iii) $\Rightarrow (i)$;
Let $P \longleftrightarrow T$ i.e $PT = TP$. To show that M reduces T, i.e $M, M^{\perp}$ are both T- invariant.
Let $x \in M$ therefore $Px = x$, since $PT = TP$ we have, $PTx = TPx$, therefore, $P(Tx) = Tx$, therefore $Tx \in M$ (Note: $\mathfrak{R}_P = M$).

Thus $x \in M$ implies $Tx \in M$, i.e M is T-invariant. Let $y \in M^{\perp}$. Then $Py = \overline{0}$. Since $PT = TP$, so $PTy = TPy = T(Py) = T(\overline{0}) = \overline{0}$.Therefore $P(Ty) = \overline{0}$ which implies $Ty \in M^{\perp}$. Thus $y \in M^{\perp} \Rightarrow Ty \in M^{\perp}$ i.e $M^{\perp}$ is T-invariant.Thus M reduces T.

□

Remark 10. *Thus the statement;M reduces T" can be given an equivalent version (algebraic or operator theory) as,$T \longleftrightarrow P$"(where P is orthogonal projector on H onto M)*

Definition 4. Let H be a Hilbert space and $T \in B(H)$. We say that an orthogonal projector P reduces T if $P \longleftrightarrow T$ (This is equivalent to saying that M reduces T where $M = \mathfrak{R}_p$).

We now give another equivalent version of an orthogonal projector;

Proposition 7. *Let H be a Hilbert space and $P \in B(H)$. Then P is an orthogonal projector if and only if $P^2 = P$ and $\|P\| \leqslant 1$.*

Proof. If P is an orthogonal projector, we have seen from proposition 1 that $P^2 = P$ and $\|P\| \leq 1$. Conversely let $P^2 = P$ and $\|P\| \leq 1$. Let $\Re_P = M$.From the idempotency of P we have already seen that M is a closed linear subspace of H and $x \in M$ implies $Px = x$.(see proposition 2). Let $x \in H$. We can write

$$x = Px + (x - Px)\,. \tag{5}$$

Consider the element $x - Px$. Now

$$P(x - Px) = Px - P(Px) = Px - P^2x = Px - Px = \bar{0} \text{ (since } P^2 = P)$$

Hence for any $x \in H, x - Px \in \eta_P = N$ (say). Since $P \in B(H)$, η_P i.e N is a closed linear subspace. So in (5) $Px \in M$ and $(x - Px) \in N$. In particular, if $x \in N^\perp$, then using (5) we can write

$$x = Px + y \text{ where } y \in N\ .$$

Therefore $Px = x - y$ where $\langle x, y\rangle = 0$ (note $x \in N^\perp, y = (x - Px) \in N\big)$. Now

$$\|x\|^2 \geq \|Px\|^2 \text{ (since } \|P\| \leq 1) = ||x - y\,\|^2 = \|\,x\,\|^2 + \|\,y\|^2$$

Therefore $\|y\|^2 = 0$ i.e $y = \bar{0}$
Thus $x = Px$ i.e $x \in M = \Re_P$.Thus

$$N^\perp \subseteq M \tag{6}$$

Conversely suppose $x \in M$. Then Since $H = N \oplus N^\perp$ (Projection theorem), we can write $x = x' + x''$ where $x' \in N$ and $x'' \in N^\perp$.So

$$Px = Px' + Px''.$$

But $Px' = \bar{0}$ (for $x' \in N = \eta_p$. Therefore $Px = Px'' = x''$ (since $x'' \in N^\perp \subseteq M$ by (6) therefore $Px'' = x''$). Thus $Px = x'' \in N^\perp$. i.e $x = x'' \in N^\perp$. So

$$M \subseteq N^\perp \tag{7}$$

From (6) and (7) we get $M = N^\perp$. Therefore $M^\perp = \left(N^\perp\right)^\perp = N$ since N is a closed linear subspace. Therefore

$$\Re_P = M \text{ and } \eta_P = N = M^\perp$$

□

These two show that P is an Orthogonal projector.

3 Conclusion

Let M be a closed linear subspace of a Hilbert space H. By projection theorem $H = M \oplus M^\perp$. For a $P \in B(H)$ we have shown that the following statements are equivalent:

i) P is an orthogonal projector such that $\Re_P = M$ and $\eta_P = M^\perp$.
ii) P is self-adjoint and idempotent.
iii) P is idempotent and $\|P\| \leq 1$.

This gives a complete operator characterization of orthorgonal projectors. If $T \in B(H)$ then T-invariance of M is essentially a geometric concept involving a linear subspace M and its image. We have shown that this geometric concept on invariance can be translated into a purely algebraic concept involving operators with perfect equivalence. The statement M reduces T" can be given an equivalent version (algebraic or operator theory) as T commutes with P.

An additional observation is that orthoprojectors are the simplest self-adjoint elements of $B(H)$ (their restrictions to their range being identity mappings, that is, if P is an orthoprojector on H with range M, then $P|_M =$ identity on M) and their importance lies in the fact that every bounded (and even unbounded) self-adjoint operator in H can be built up in some sense from orthoprojectors. This is indeed the central theme and result of the spectral theory of self-adjoint operators and the very idea, in an abstract sense, of expressing an operator H (bounded or not) in some sense in terms of orthoprojectors, is the basic philosophy in the evolution of the spectral theory of linear operators in a Hilbert space.

Competing Interests

Authors have declared that no competing interests exist.

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