Research Article

Axial Symmetry Cosmological Constant Vacuum Solution of Field Equations with a Curvature Singularity, Closed Time-Like Curves, and Deviation of Geodesics

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In this paper, we present a type D, nonvanishing cosmological constant, vacuum solution of Einstein's field equations, extension of an axially symmetric, asymptotically flat vacuum metric with a curvature singularity. The space-time admits closed time-like curves (CTCs) that appear after a certain instant of time from an initial space-like hypersurface, indicating it represents a time-machine space-time. We wish to discuss the physical properties and show that this solution can be interpreted as gravitational waves of Coulomb-type propagate on anti-de Sitter space backgrounds. Our treatment focuses on the analysis of the equation of geodesic deviations.

1. Introduction

Closed time-like curves constitute one of the most intriguing aspects of general relativity. The first solution of the field equations admitting closed time-like curves (CTCs) is the Gödel rotating Universe [1]. It represents a rotating universe and is axially symmetric, given by

$$ds^2 = dr^2 + dz^2 + \left(\sinh^2 r - \sinh^4 r\right)d\theta^2 + 2\sqrt{2}\,\sinh^2 r d\theta dt - dt^2 \tag{1}$$

The coordinates are in the ranges $0 \leq r < \infty$, $-\infty < z < \infty$, and $-\infty < t < \infty$, and θ is periodic. For some $r > r_0$, the metric function, $g_{\theta\theta} = \sinh^2 r - \sinh^4 r$, becomes negative. The circle defined by $r > r_0$ and $t = 0 = z$ will be time-like everywhere. This condition is fulfilled when $r > r_0 = \ln\left(1 + \sqrt{2}\right)$ which is the condition for the existence of CTCs in the Gödel space-time because one of the coordinates $\theta \in [0, 2\pi]$ is periodic. The next one is the van Stockum space-time [2], which predates the Gödel solution and was shown later to have CTCs [3]. Examples of space-time admitting CTCs including

NUT-Taub metric [4–7], Kerr and Kerr-Newman black hole solution [8–10], Gott time-machine [11], Grant space-time [12], Krasnikov tube [13], Bonnor's metrics [14–19], and others [20–36]. Space-time with causality violating curves is classified as either eternal or true time-machine space-times. In eternal time machine space-time case, CTCs always preexist. In this category would be [1] or [2] (see also, Refs. [23, 25, 27, 32, 36]). A true time machine space-time is the one in which CTCs evolve at a particular instant of time from an initial space-like hypersurface in a causally well-behaved manner satisfying all the energy conditions with known type of matter fields. In this category, the Ori time machine space-time [37] is considered to be most remarkable. But the matter sources satisfying all the energy conditions are of unknown type in this space-time. Most of the time machine models suffer from one or more drawbacks. For space-time admitting CTCs, the matter-energy sources must be realistic, that is, the stress-energy tensor must be of a known type of matter fields, which satisfy all the energy conditions. Many space-time model, for examples, traversable wormholes [38, 39] and warp drive models [40–43] violate the weak energy condition (WEC), which states that $T_{\mu\nu}U^\mu U^\nu \geq 0$ for a time-like

tangent vector field U^μ, that is, the energy-density must be nonnegative. Some other space-times admitting CTCs violate the strong energy condition (SEC) (e.g., Refs. [44–47]), which states that $(T_{\mu\nu} - 1/2g_{\mu\nu}T)U^\mu U^\nu \geq 0$. In addition, some solutions do not admit a partial Cauchy surface (initial space-like hypersurface) (e.g., Refs. [1, 48]) and/or CTCs come from infinity (e.g., Refs. [11, 12]). In addition, there is a curvature singularity in some solutions admitting CTCs [3, 35, 48–53].

In literature, only a handful of solutions of Einstein's field equations with the stress-energy tensor in [1, 33, 34] and type N Einstein space-time in [30] have a negative cosmological constant. In this work, we try to construct a type D Einstein space-time with a negative cosmological constant which was not studied earlier. The cosmological constant plays a vital role in explaining the dynamics of the universe. A tiny positive cosmological constant neatly explains the late-time accelerated expansion of the universe. Indeed, our universe is observed to be undergoing a de Sitter (dS) type expansion in the present epoch. For a negative cosmological constant, space-time is labelled as an anti-de Sitter (AdS) space. The AdS space has been a subject of intense study in recent times on account of the celebrated AdS/CFT correspondence [54], which provides a link between a quantum theory of gravity on an asymptotically AdS space and a lower-dimensional conformal field theory (CFT) on its boundary.

2. Review of a Type D Vacuum Space-Time with a Curvature Singularity and CTCs [51]

In Ref. [51], a type D axially symmetric, asymptotically flat vacuum solution of the field equations with zero cosmological constant, was constructed. This vacuum metric is as follow

$$ds^2 = -\cosh t \coth t \sinh^2 r dt^2 + \cosh^2 r \sinh r dr^2$$
$$+ \operatorname{csch} r dz^2 + \sinh^2 r \left(2\sqrt{2} \cosh t dt d\phi - \sinh t d\phi^2\right). \tag{2}$$

After doing a number of transformations into the above metric, we arrive at the following

$$ds^2 = \cosh^2 r \sinh r dr^2 + \operatorname{csch} r dz^2 - \sinh^2 r \left(2dt d\phi + t d\phi^2\right). \tag{3}$$

The Kretschmann scalar of the above metric is

$$K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{12}{\sinh^6 r}. \tag{4}$$

For constant r, z, the metric (3) reduces to conformal Misner metric in 2D

$$ds^2 = \Omega\left(-2dt d\phi - t d\phi^2\right), \tag{5}$$

where $\Omega = \sinh^2 r$ is the conformal factor.

In the context of CTCs, the Misner space metric in 2D is interesting because CTCs appear after a certain instant of time from causally well-behaved conditions. The metric for the Misner space in 2D [55] is given by

$$ds_{\text{Mis}}^2 = -2dT dX - T dX^2, \tag{6}$$

where $-\infty < T < \infty$ but the coordinate X is periodic locally. The metric (6) is regular everywhere as $\det g = -1$ including at $T = 0$. The curves $T = T_0$, where T_0 is a constant, are closed since X is periodic. The curves $T < 0$ are spacelike, $T > 0$ are time-like, while the null curves $T = 0$ form the chronology horizon. The second type of curves, namely, $T = T_0 > 0$, are closed time-like curves. Therefore, the metric (2) or (3) is a four-dimensional generalization of Misner space in curved space-time. Note that the above space-time is the vacuum solution of field equations, a Ricci flat, that is, $R_{\mu\nu} = 0$. Li [56] constructed a Misner-like AdS space-time, a time-machine model. Levanony and Ori [57] constructed a three-, four-dimensional generalization of flat Misner space metric.

In this paper, we extend the above Ricci flat space-time (2) to the Einstein space-times of Petrov type D, which satisfy the following conditions

$$R_{\mu\nu} = \Lambda g_{\mu\nu}, R = 4\Lambda, \Lambda < 0 \text{ or } \Lambda > 0. \tag{7}$$

It is an anti-de Sitter-like space if $\Lambda < 0$ and de Sitter like if $\Lambda > 0$. The extended space-time satisfies all the basic requirements (see details in Ref. [30]) for a time machine space-time except one, that is, this new model is not free from curvature singularity.

3. Analysis of a Cosmological Constant Vacuum Space-Time

Consider the following line element, a modification of the metric (2) given by

$$ds^2 = \sinh^2 r \left(-\cosh t \coth t dt^2 + 2\sqrt{2} \cosh t dt d\phi - \sinh t d\phi^2\right)$$
$$+ \frac{dr^2}{(\alpha \operatorname{csch} r \operatorname{sech}^2 r + \beta^2 \tanh^2 r)} + \left(\alpha \operatorname{csch} r + \beta^2 \sinh^2 r\right)dz^2. \tag{8}$$

Here, α is a positive constant, and β is real. The coordinates are labelled $x^0 = t$, $x^1 = r$, $x^2 = \phi$, and $x^3 = z$. The ranges of the coordinates are

$$-\infty < t < \infty, 0 \leq r < \infty, -\infty < z < \infty, \tag{9}$$

and ϕ is a periodic coordinate $\phi \sim \phi + \phi_0$, with $\phi_0 > 0$. The metric is Lorentzian with signature $(-, +, +, +)$ and the determinant of the corresponding metric tensor $g_{\mu\nu}$,

$$\det g = -\cosh^2 r \sinh^4 r \cosh^2 t. \tag{10}$$

Now, we have evaluated the Ricci tensor $R_{\mu\nu}$ of the space-time (8) as follows:

$$R_{tt} = 3\beta^2 \cosh t \coth t \sinh^2 r,$$

$$R_{t\phi} = R_{\phi t} = -3\sqrt{2}\beta^2 \cosh t \sinh^2 r,$$

$$R_{rr} = -3\beta^2 \left(\frac{\cosh^2 r}{\alpha \operatorname{csch} r + \beta^2 \sinh^2 r} \right), \tag{11}$$

$$R_{\phi\phi} = 3\beta^2 \sinh^2 r \sinh t,$$

$$R_{zz} = -3\beta^2 \left(\alpha \operatorname{csch} r + \beta^2 \sinh^2 r \right).$$

The Ricci scalar is given by

$$R_\mu^\mu = R = -12\beta^2. \tag{12}$$

Using the metric tensor components of the above space-time, we have found that the Ricci tensor

$$R_{\mu\nu} = -3\beta^2 g_{\mu\nu} (\mu, \nu = 0, 1, 2, 3), \tag{13}$$

and the Einstein tensor $G_{\mu\nu}$ are

$$G_\nu^\mu = 3\beta^2 \operatorname{diag}(1, 1, 1, 1). \tag{14}$$

From the Einstein's field equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ and from eq. (14), we have

$$3\beta^2 = -\Lambda \Rightarrow \beta = \pm\sqrt{-\frac{\Lambda}{3}}, \Lambda < 0. \tag{15}$$

Thus, from the above analysis, it is clear that the space-time considered by (8) is an example of the class of Einstein space of anti-de Sitter-type and satisfies eq. (7) for a negative cosmological constant. We have shown later that the space-time possesses a curvature singularity at $r \to 0$.

An interesting property of the metrics (8) is that it reduces to 2D Misner space metric [55] for constant r, z. For that, we do the following transformations

$$t \to \sinh^{-1} t, \phi \to \phi + \left(\sqrt{2} + 1 \right) \ln t, \tag{16}$$

into the metric (8) (replacing $\beta^2 \to -\Lambda/3$), we arrive at the following line element

$$ds^2 = \sinh^2 r \left(-2dtd\phi - td\phi^2 \right) + \frac{\cosh^2 r dr^2}{\left(\alpha \operatorname{csch} r - (\Lambda/3) \sinh^2 r \right)} \tag{17}$$

$$+ \left(\alpha \operatorname{csch} r - \frac{\Lambda}{3} \sinh^2 r \right) dz^2. \tag{18}$$

For constant $r = r_0 > 0$ and $z = z_0$, the metric (17) becomes

$$ds_{\text{conf}}^2 = \sinh^2 r \left(-2dtd\phi - td\phi^2 \right) = \Omega ds_{\text{Mis}}^2, \tag{19}$$

a conformal Misner space metric in 2D where Ω is the conformal factor. Therefore, the space-time admits CTC for $t = t_0 > 0$ similar to the Misner space discussed earlier.

We check whether the CTCs evolve from an initially space-like $t = $ constant hypersurface (and thus t is a time coordinate). This is determined by calculating the norm of the vector $\nabla_\mu t$ [37] (or alternately from the value of g^{tt} in the inverse metric tensor $g^{\mu\nu}$). A hypersurface $t = $ constant is space-like when $g^{tt} < 0$ at $t < 0$, time-like when $g^{tt} > 0$ for $t > 0$, and null $g^{tt} = 0$ for $t = 0$. For the metric (8), we have

$$\nabla_\mu t \nabla^\mu t = g^{tt} = \frac{\sinh t}{\sinh^2 r \, \cosh^2 t}. \tag{20}$$

Thus, a hypersurface $t = $ constant is spacelike for $t < 0$, time-like for $t > 0$, and null at $t = 0$. We restrict our analysis to $r > 0$; otherwise, no CTCs will be formed. Thus, the space-like $t = $ constant < 0 hypersurface can be chosen as initial hypersurface over which initial data may be specified. There is a Cauchy horizon at $t = t_0 = 0$ called the chronology horizon, which separates the causal past and future in a past-directed and future-directed manner. Hence, the space-time evolves from a partial Cauchy surface (i.e., initial space-like hypersurface) in a causally well-behaved, up to a moment, i.e., a null hypersurface $t = t_0 = 0$ and the formation of CTCs takes place from causally well-behaved initial conditions. The evolution of CTC is thus identical to the case of the Misner space.

That the space-time represented by (8) satisfies the requirements of axial symmetry is clear from the following. Consider the Killing vector $\eta = \partial_\phi$ having the normal form

$$\eta^\mu = (0, 0, 1, 0). \tag{21}$$

Its covector form

$$\eta_\mu = \sinh^2 r \left(\sqrt{2} \cosh t, 0, -\sinh t, 0 \right). \tag{22}$$

The vector (22) satisfies the Killing equation $\eta_{\mu;\nu} + \eta_{\nu;\mu} = 0$. The space-time is axial symmetry if the norm of the Killing vector η^μ vanishes on the axis i.e., at $r = 0$ (see [58, 59] and references therein). In our case

$$X = |\eta_\mu \eta^\mu| = |g_{\phi\phi}| = |-\sinh t \sinh^2 r| \to 0, \tag{23}$$

as $r \to 0$.

The metric has a curvature singularity at $r = 0$. We find that the Kretschmann scalar is

$$K = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{8\Lambda^2}{3} + \frac{12\alpha^2}{\sinh^6 r}. \tag{24}$$

We can see that the scalar curvature diverges at $r \to 0$, which indicates that the space-time possesses a curvature singularity. In addition, the Kretschmann scalar becomes $K \to 8\Lambda^2/3$ for $r \to \infty$, indicating that the metric (8) is asymptotically anti-de Sitter-like space radially [60].

3.1. Classification and Physical Interpretation of the Space-Times.
Here, we first classify the space-time according to the Petrov classification scheme and then analyze the effect of local fields of the solution. We construct a set of null tetrad $(k, l, m, \bar{m})$ [61] for the space-time (8). Explicitly, these covectors are

$$k_\mu = \frac{\sinh r}{\sqrt{2}} \left(\frac{\cosh t}{\sqrt{\sinh t}}, 0, \left(-\sqrt{2}+1\right)\sqrt{\sinh t}, 0 \right), \quad (25)$$

$$l_\mu = \frac{\sinh r}{\sqrt{2}} \left(\frac{\cosh t}{\sqrt{\sinh t}}, 0, -\left(\sqrt{2}+1\right)\sqrt{\sinh t}, 0 \right), \quad (26)$$

$$m_\mu = \frac{1}{\sqrt{2}} \left(0, \frac{\cosh r}{\sqrt{\alpha \operatorname{csch} r - (\Lambda/3)\sinh^2 r}}, 0, i\sqrt{\alpha \operatorname{csch} r - \frac{\Lambda}{3}\sinh^2 r} \right), \quad (27)$$

$$\bar{m}_\mu = \frac{1}{\sqrt{2}} \left(0, \frac{\cosh r}{\sqrt{\alpha \operatorname{csch} r - (\Lambda/3)\sinh^2 r}}, 0, -i\sqrt{\alpha \operatorname{csch} r - \frac{\Lambda}{3}\sinh^2 r} \right). \quad (28)$$

The set of null tetrad above is such that the metric tensor for the line element (8) can be expressed as

$$g_{\mu\nu} = -k_\mu l_\nu - l_\mu k_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu. \quad (29)$$

The vectors (25), (26), (27), and (28) are null vector and orthogonal, except for $k_\mu l^\mu = -1$ and $m_\mu \bar{m}^\mu = 1$.

We calculate the five Weyl scalars, of these only

$$\Psi_2 = C_{\mu\nu\rho\sigma} k^\mu m^\nu \bar{m}^\rho l^\sigma = \frac{\alpha}{2\sinh^3 r}, \quad (30)$$

is nonvanishing, while the rest are vanish. Thus, the metric is clearly of type D in the Petrov classification scheme.

We set up an orthonormal frame $e_{(a)} = \{e_{(0)}, e_{(1)}, e_{(2)}, e_{(3)}\}$, $e_{(a)} \cdot e_{(b)} \equiv e^\mu_{(a)} e^\nu_{(b)} g_{\mu\nu} = \eta_{ab} = \operatorname{diag}(-1,+1,+1,+1)$, which consists of three space-like unit vectors $e_{(i)}$, $i = 1, 2, 3$ and one time-like vector $e_{(0)}$ [62]. Notations are such that small Latin indices are raised and lowered with Minkowski metric η^{ab}, η_{ab}, and Greek indices are raised and lowered with metric tensor $g^{\mu\nu}$, $g_{\mu\nu}$. The dual basis is $e^{(i)} = e_{(i)}$ and $e^{(0)} = -e_{(0)}$. These frame components in terms of tetrad vector can be expressed as

$$k = \frac{1}{\sqrt{2}} \left(e_{(0)} + e_{(2)} \right), l = \frac{1}{\sqrt{2}} \left(e_{(0)} - e_{(2)} \right), m = \frac{1}{\sqrt{2}} \left(e_{(1)} + i e_{(3)} \right). \quad (31)$$

In order to analyze the effect of local gravitational fields of these solutions, we have used the equations of geodesic deviation [25, 33, 52, 63–66] which in terms of orthonormal frame $e_{(a)}$ are

$$\ddot{Z}^{(i)} = -R^{(i)}_{(0)(j)(0)} Z^{(j)}, i, j = 1, 2, 3, \quad (32)$$

where $e_{(0)} = u$ is a time-like four-velocity vector of the free test particles. We set here $Z^{(0)} = 0$ such that all test particles are synchronized by the proper time. From the standard definition of the Weyl tensor and the Einstein's field equation for zero the stress-energy tensor, we get (see Eq. (4) in [66])

$$R_{(i)(0)(j)(0)} = C_{(i)(0)(j)(0)} - \frac{\Lambda}{3}\delta_{ij}, \quad (33)$$

where $C_{(i)(0)(j)(0)} \equiv e^\mu_{(i)} u^\nu e^\rho_{(j)} u^\sigma C_{\mu\nu\rho\sigma}$ are the components of the Weyl tensor.

The only nonvanishing Weyl scalars are given by (30) so that

$$C_{(1)(0)(1)(0)} = -\Psi_2 = C_{(3)(0)(3)(0)}, C_{(2)(0)(2)(0)} = 2\Psi_2. \quad (34)$$

Therefore, the equations of geodesic deviation (32) take the following form

$$\ddot{Z}^{(1)} = -R^{(1)}_{(0)(j)(0)} Z^{(j)} = -\left(C_{(1)(0)(1)(0)} - \frac{\Lambda}{3} \right) Z^{(1)}$$
$$= \left(\Psi_2 + \frac{\Lambda}{3} \right) Z^{(1)}, \quad (35)$$

$$\ddot{Z}^{(2)} = -R^{(2)}_{(0)(j)(0)} Z^{(j)} = -\left(C_{(2)(0)(2)(0)} - \frac{\Lambda}{3} \right) Z^{(2)}$$
$$= \left(-2\Psi_2 + \frac{\Lambda}{3} \right) Z^{(2)}, \quad (36)$$

$$\ddot{Z}^{(3)} = -R^{(3)}_{(0)(j)(0)} Z^{(j)} = -\left(C_{(3)(0)(3)(0)} - \frac{\Lambda}{3} \right) Z^{(3)}$$
$$= \left(\Psi_2 + \frac{\Lambda}{3} \right) Z^{(3)}. \quad (37)$$

In the limit $\alpha \to 0$, all the Weyl scalars including Ψ_2 vanishes. In this limit, the space-time (8) becomes anti-de Sitter (AdS) space. So the equations of geodesic deviation (35) in this limit reduces to

$$\ddot{Z}^{(i)} = \frac{\Lambda}{3} Z^{(i)}, \quad (38)$$

with the solutions

$$Z^{(i)} = a_i \tau + b_i \text{ for } \Lambda = 0,$$

$$Z^{(i)} = a_i \cos\left(\sqrt{-\frac{\Lambda}{3}}\tau\right) + b_i \sin\left(\sqrt{-\frac{\Lambda}{3}}\tau\right) \text{ for } \Lambda < 0,$$

$$(39)$$

where a_i, b_i, $i = 1, 2, 3$ are the arbitrary constants.

Again, in the limit $\Lambda \to 0$, that is, $\beta \to 0$, the only non-vanishing Weyl scalars is Ψ_2 given by (30). The space-time (8) reduces to type D vacuum space-time of zero cosmological constant with a curvature singularity which we discussed, in detail in Ref. [51]. In this limit ($\Lambda \to 0$), the equations of geodesic deviation (35) becomes

$$\ddot{Z}^{(1)} = \Psi_2 Z^{(1)}, \ddot{Z}^{(2)} = -2\Psi_2 Z^{(2)}, \ddot{Z}^{(3)} = \Psi_2 Z^{(3)}, \quad (40)$$

with the solutions

$$Z^{(1)} = c_1 \cosh\left(\sqrt{\Psi_2}\tau\right) + d_1 \sinh\left(\sqrt{\Psi_2}\tau\right),$$

$$Z^{(2)} = c_2 \cos\left(\sqrt{2\Psi_2}\tau\right) + d_2 \sin\left(\sqrt{2\Psi_2}\tau\right), \quad (41)$$

$$Z^{(3)} = c_3 \cosh\left(\sqrt{\Psi_2}\tau\right) + d_3 \sinh\left(\sqrt{\Psi_2}\tau\right),$$

where c_i, d_i, $i = 1, 2, 3$ are the arbitrary constants and $\Psi_2 \neq 0$.

4. Summary and Future Work

In this paper, we generalize a Ricci flat space-time [51] to the case of nonvanishing cosmological constant solution in four-dimensional curved space-time, still represent vacuum solutions of the Einstein's field equations are the generalization of 2D Misner space metric. By introducing a cosmological constant term into the metric components in the metric [51], we have seen that for $r = r_0$, and $z = z_0$, where r_0, z_0 are constants, these space-times reduce to 2D conformal Misner space geometry. As discussed in Section 2, the Misner space metric admits CTCs which appear after a certain instant of time from causally well-behaved conditions. Thus, the presented metrics as well as the one studied in [51] evolve CTC from an initial space-like hypersurface at a certain instant of time. Though causality violating space-times have been studied extensively in the literature, few of them belongs to true time-machine space-time (e.g., [24–26, 28, 31, 33, 34, 37]), and others in (e.g., [26, 51–53]) are lacking one or more basic requirements for a true time-machine space-time. In addition, many time-machine models mentioned in the introduction violate one or more the energy condition. Our space-time is the vacuum solution of Einstein's field equations of nonzero cosmological constant. So all the energy conditions are automatically satisfied, and the modified metrics would represent true time-machine space-time but lacking the property of being free from curvature singularity. Furthermore, we have analyzed the space-time and discussed their physical properties. It was demonstrated that these space-times can be understood as gravitational waves of Coulomb-types which propagate on anti-de Sitter backgrounds. A positive cosmological constant (Λ) plays an important role in explaining the dynamics of the universe. But in our case, however, it is negative. One can use this modified space-time as a model to study the quantum gravity in connection to the quantum field theory. The dynamic stability of the modified space-times is beyond the scope of this work. Our motivation to further study this problem is to construct a space-time metric which satisfies all criteria for a true time-machine, like obeying the energy conditions, realistic or known types of matter sources, singularity-free and evolves CTCs from an initial space-like hypersurface in a causally well-behaved manner after a certain instant of time.

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